Journal of Modern Materials
ISSN: 2456-4834
Volume 9, Issue 1, pp. 36-49, 2022
doi:https://doi.org/10.21467/jmm.9.1.36-49
RESEARCH ARTICLE

Applying a Set of Orthogonal Basis Functions in Numerical Solution of Hallén's Integral Equation for Dipole Antenna of Perfectly Conducting Material

Saeed Hatamzadeh [1]*, Zahra Masouri [2]

[1]Department of Electrical Engineering, Islamshahr Branch, Islamic Azad University, Tehran, Iran
[2]Department of Mathematics, Islamshahr Branch, Islamic Azad University, Tehran, Iran

* Corresponding author email: s.hatamzadeh@yahoo.com

Received: 09 April 2022 / Revised: 20 May 2022 / Accepted: 26 September 2022 / Published: 01 November 2022

ABSTRACT

The focus of this paper is on solving Hallén's integral equation for a diploe antenna of perfectly conducting material. A special representation of orthogonal triangular basis functions is used to implement an effective numerical method for solving this equation. The Hallén's formulation is treated in detail and illustrative computations are given for current distributions and radiation patterns.

Keywords: Perfectly conducting material; Hallén's integral equation; Orthogonal triangular functions.

1 Introduction

Integral equation approach has a wide variety of applications in material science. Some instances of such applications are reviewed here. Chai *et al.* [1] studies the stress intensity factors and their correlations with bimaterial parameters for the interface planar cracks by hypersingular integral equations. A new bimaterial parameter γ is defined, and the correlations of the stress intensity factors of interface planar crack with two parameters ε and γ are proved. The relative displacement fundamental function is proposed based on the crack's peripheral equation. In [2], a combined Laplace transform and boundary element method is used to find numerical solutions to problems of another class of anisotropic functionally graded materials which are governed by a variable coefficients parabolic equation. A transformation is used to reduce the variable coefficients equation to a constant coefficients equation, which is then transformed into a boundary-only integral equation. In [3], the new system of hypersingular integral equations (HSIEs) for the thermally insulated inclined cracks and thermally insulated circular arc cracks subjected to remote shear stress in bonded dissimilar materials is formulated by using the modified complex potentials (MCPs) function method with the continuity conditions of the resultant force, displacement and heat conduction functions. This new system of HSIEs is derived from the elasticity problem and heat conduction problem by using crack opening displacement (COD) function and temperature jump along the crack faces. In [4], the convergence of an iterative method called Adomian decomposition method is analyzed to solve the Fuzzy Volterra Integral Equations (FVIE) with time lag, the uniqueness and convergence of the method are equivalent. In [5], a spectral formulation of the boundary integral equation method for antiplane problems is presented. The boundary integral equation method relates the slip and the shear stress at an interface between two half-planes. It involves evaluating a space-time convolution of the shear stress or the slip at the interface. In the spectral formulation, the convolution with respect to the spatial coordinate is performed in the spectral domain. This leads to greater numerical efficiency. In [6], the static interaction between an eccentricly loaded rectangular rigid foundation and layered transversely isotropic soils is investigated. Firstly, the solution of the layered transversely isotropic soils is derived using the analytical layer element method. After decomposing the deformation forms of the rigid rectangular foundation under eccentric loads, the dual integral equations are established through mixed boundary conditions. Then, the Jacobi orthogonal polynomial and the Bessel function are employed to solve the above dual integral equations, and the static response solution of the rigid rectangular foundation subjected to eccentric loads is obtained through superposition.

Hallén's equation is a first kind Fredholm integral equation [7]. For solving integral equations of the first kind, several numerical approaches have been proposed [8]. These numerical methods often use the basis functions and transform the integral equation to a linear system that can be solved by direct or iterative methods [8]. It is important in these methods to select an appropriate set of basis functions so that the approximate solution of integral equation has a good accuracy.

In this paper, we use a special representation of orthogonal triangular basis functions to implement a numerical method for solving Hallén's integral equation for a dipole antenna of perfectly conducting material. Using this method, the integral equation reduces to a linear system of algebraic equations. Solving this system gives an approximate solution for the problem.

First of all, we review the special representation of triangular functions. Then, a numerical method is formulated for solving Fredholm integral equation of the first kind based on the mentioned basis functions. Finally, Hallén's integral equation is solved via the presented method, and illustrative computations for current distributions and radiation patterns are given to complete the procedure.

2 Triangular functions

A special representation of triangular functions has been introduced by A. Deb et al. [9] as a set of orthogonal functions.

Two m-sets of triangular functions (TFs) are defined over the interval [0,T) as [9]

$$
\begin{aligned}
T1_i(t) &= \begin{cases} 1 - \frac{t-ih}{h}, & ih \le t < (i+1)h, \\ 0, & \text{otherwise}, \end{cases} \\
T2_i(t) &= \begin{cases} \frac{t-ih}{h}, & ih \le t < (i+1)h, \\ 0, & \text{otherwise}, \end{cases}
\end{aligned} \tag{1}
$$

where, $i = 0, 1, \ldots, m-1$, with a positive integer value for m. Also, consider $h = T/m$, and $T1_i$ as the ith left-handed triangular function and $T2_i$ as the ith right-handed triangular function.

These functions are orthogonal [9], so

$$\begin{aligned}
\int_0^1 T1_i(t)T1_j(t)\,\mathrm{d}t &= \begin{cases} \frac{h}{3} & i = j, \\ 0 & i \neq j, \end{cases} \\
\int_0^1 T2_i(t)T2_j(t)\,\mathrm{d}t &= \begin{cases} \frac{h}{3} & i = j, \\ 0 & i \neq j. \end{cases}
\end{aligned} \tag{2}$$

Now, consider the first m terms of left-handed triangular functions and the first m terms of right-handed triangular functions and write them concisely as m-vectors

$$\begin{aligned}
\mathbf{T1}(t) &= [T1_0(t), T1_1(t), \ldots, T1_{m-1}(t)]^t, \\
\mathbf{T2}(t) &= [T2_0(t), T2_1(t), \ldots, T2_{m-1}(t)]^t,
\end{aligned} \tag{3}$$

where, $\mathbf{T1}(t)$ and $\mathbf{T2}(t)$ are called left-handed triangular functions (LHTF) vector and right-handed triangular functions (RHTF) vector, respectively.

The expansion of a function $f(t)$ with respect to TFs, may be compactly written as

$$\begin{aligned}
f(t) &\simeq \sum_{i=0}^{m-1} c_i T1_i(t) + \sum_{i=0}^{m-1} d_i T2_i(t) \\
&= \mathbf{c}^T \mathbf{T1}(t) + \mathbf{d}^T \mathbf{T2}(t),
\end{aligned} \tag{4}$$

where, c_i and d_i are constant coefficients with respect to $T1_i$ and $T2_i$ for $i = 0, 1, \ldots, m-1$, respectively.

Above coefficients can be determined by sampling $f(t)$ such that

$$\begin{aligned}
c_i &= f(ih), \\
d_i &= f((i+1)h), \quad \text{for } i = 0, 1, \ldots, m-1.
\end{aligned} \tag{5}$$

But the optimal representation of $f(t)$ can be obtained if the coefficients c_i and d_i are determined from the following two equations [9]:

$$\begin{aligned}
\int_{ih}^{(i+1)h} f(t)T1_i(t)\,\mathrm{d}t &= c_i \int_{ih}^{(i+1)h} [T1_i(t)]^2\,\mathrm{d}t + d_i \int_{ih}^{(i+1)h} [T1_i(t)T2_i(t)]\,\mathrm{d}t, \\
\int_{ih}^{(i+1)h} f(t)T2_i(t)\,\mathrm{d}t &= c_i \int_{ih}^{(i+1)h} [T1_i(t)T2_i(t)]\,\mathrm{d}t + d_i \int_{ih}^{(i+1)h} [T2_i(t)]^2\,\mathrm{d}t.
\end{aligned} \tag{6}$$

Note that

$$\int_{ih}^{(i+1)h} [T1_i(t)T2_i(t)]\,\mathrm{d}t = \frac{h}{6}. \tag{7}$$

From Eqs. (6) and Eq. (7) coefficients c_i and d_i for $i = 0, 1, \ldots, m-1$ can be easily computed.

It is clear that for piecewise linear functions, optimal and non-optimal representations are identical.

3 Numerical method to solve first kind Fredholm integral equation using triangular functions

In this section, the triangular functions are used for formulation of a numerical method to solve Fredholm integral equation of the first kind. For this purpose, the definition of triangular functions is extended over arbitrary interval $[a, b)$.

Consider the following Fredholm integral equation of the first kind:

$$\int_a^b k(s,t)x(t)\,\mathrm{d}t = y(s), \tag{8}$$

where, $k(s,t)$ and $y(s)$ are known functions but $x(t)$ is unknown. Moreover, $k(s,t) \in \mathcal{L}^2([a,b) \times [a,b))$ and $y(s) \in \mathcal{L}^2([a,b))$. Approximating the function $x(s)$ with respect to triangular functions by (4) gives

$$x(s) \simeq \mathbf{c}^T \mathbf{T1}(s) + \mathbf{d}^T \mathbf{T2}(s), \tag{9}$$

such that the m-vectors $\mathbf{c}$ and $\mathbf{d}$ are TFs coefficients of $x(s)$ that should be determined.

Substituting Eq. (9) into (8) follows:

$$\mathbf{c}^T \int_a^b k(s,t)\mathbf{T1}(t)\,\mathrm{d}t + \mathbf{d}^T \int_a^b k(s,t)\mathbf{T2}(t)\,\mathrm{d}t \simeq y(s). \tag{10}$$

Now, let s_i, $i = 0, 1, \dots, 2m-1$ be $2m$ appropriate points in interval $[a, b)$; putting $s = s_i$ in Eq. (10) follows:

$$\mathbf{c}^T \int_a^b k(s_i,t)\mathbf{T1}(t)\,\mathrm{d}t + \mathbf{d}^T \int_a^b k(s_i,t)\mathbf{T2}(t)\,\mathrm{d}t \simeq y(s_i), \qquad i = 0, 1, \dots, 2m-1, \tag{11}$$

or

$$\sum_{j=0}^{m-1} \left[c_j \int_a^b k(s_i,t)T1_j(t)\,\mathrm{d}t + d_j \int_a^b k(s_i,t)T2_j(t)\,\mathrm{d}t \right] \simeq y(s_i), \qquad i = 0, 1, \dots, 2m-1. \tag{12}$$

Now, replace $\simeq$ with $=$, hence Eq. (12) is a linear system of $2m$ algebraic equations for $2m$ unknown components $c_0, c_1, \dots, c_{m-1}$ and $d_0, d_1, \dots, d_{m-1}$. So, an approximate solution $x(s) \simeq \mathbf{c}^T \mathbf{T1}(s) + \mathbf{d}^T \mathbf{T2}(s)$, is obtained for Eq. (8).

4 Hallén's integral equation modeling for a dipole antenna of perfectly conducting material

In this section, we focus on solving Hallén's equation.

Hatamzadeh & Masouri, *J. Mod. Mater.*; Vol. 9, Issue 1, pp: 36-49, 2022

A practical center-fed dipole antenna of perfectly conducting material usually consists of a pair of tubular perfect conductors of radius a aligned in tandem so that there is a small feeding gap at the center [7], as shown in Fig. 1. The total length is $2l$. A voltage is applied across the gap, often by means of a two-wire transmission line. The resulting current distribution on the pair of tubular conductors gives rise to radiating field.

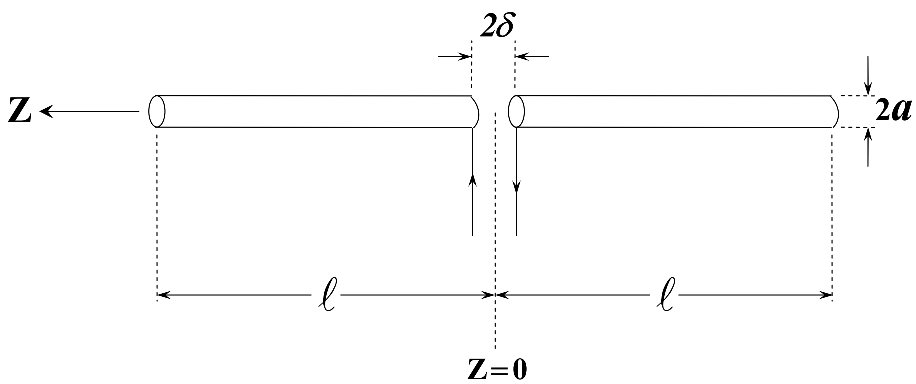

Figure 1: Cylindrical dipole antenna.

4.1 Hallén's Formulation

Referring to Fig. 1, let us assume that the length of the cylinder is much larger than its radius ($2l \gg a$) and its radius is much smaller than the wavelength ($a \ll \lambda$), so that the effects of the end faces of cylinder can be neglected. Therefore, the boundary conditions for a wire made of a material with infinite conductivity are those of vanishing total tangential **E** fields on the surface of the cylinder and vanishing current at the ends of the cylinder $\left[I_z(z = \pm l) = 0\right]$ [10]. For convenience, we assume that a constant voltage V_i is applied at the input terminals of the dipole, i.e., the *delta-gap* excitation. Under these conditions, the final form of the current integral equation is [7, 10]

$$\int_{-l}^{l} \frac{e^{-jkr}}{4\pi r} I_z(z')\,\mathrm{d}z' = C\cos kz - \frac{j\omega\epsilon_0}{2k}\sin k|z|, \tag{13}$$

where

$r = [a^2 + (z - z')^2]^{1/2}$;

$I_z(z)$, is current distribution on the cylinder;

$k = \frac{2\pi}{\lambda}$, is free space wave number;

$\epsilon_0 = 8.854 \times 10^{-12}$ F/m, is free space permittivity.

Eq. (13) is referred to as *Hallén's integral equation*. Solving this equation gives the current distribution along the dipole. Since, in this problem, the current distribution $I_z(z)$ is an even function [7], Hallén's integral equation can be rewritten in the form

$$\int_{0}^{l} G(z, z') I_z(z')\,\mathrm{d}z' = C\cos kz - \frac{j\omega\epsilon_0}{2k}\sin kz, \tag{14}$$

in which

$G(z, z') = \frac{e^{-jkr}}{4\pi r} + \frac{e^{-jkr'}}{4\pi r'}$;
$r = [a^2 + (z - z')^2]^{1/2}$;
$r' = [a^2 + (z + z')^2]^{1/2}$;
$z \in [0, l]$.

4.2 Solving Hallén's integral equation via presented approach

Now, we have the necessary tools for solving Hallén's integral equation. Applying the proposed method to solve Hallén's equation gives the current distribution along the dipole. It should be mentioned that for calculating the unknown coefficient C that appears on the right hand side of Eq. (14), the number of match points should be $2m + 1$ instead of $2m$. After calculating the current distribution we can determine the radiation pattern of dipole. The radiation pattern can be obtained of the following equation [11]:

$$f(\theta) = \int_{-l}^{l} I_z(z') e^{jkz' \cos\theta} \, \mathrm{d}z', \tag{15}$$

where, θ is observation angle. Referring to Fig. 1, we can consider θ as the θ-coordinate of spherical coordinate system.

Current distributions along the dipole for $2l = 0.4\lambda, \lambda, 1.5\lambda$ (with a specific value for a), and for $z > 0$ have been calculated and given in Figs. 2–4. The normalized radiation patterns as a function of θ are given in Figs. 5–7. Also, the radiation patterns in polar system are given in Figs. 8–10. Finally, figures 11–13 give the three-dimensional radiation patterns.

ISSN:2456-4834
Available online at journals.aijr.org

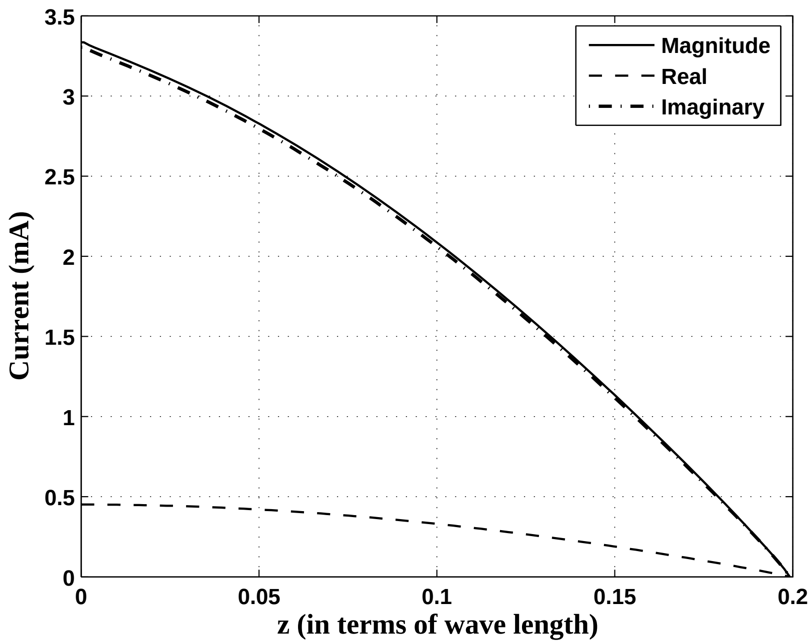

Figure 2: Current distribution for $2l = 0.4\lambda$ $a = 0.0001l$.

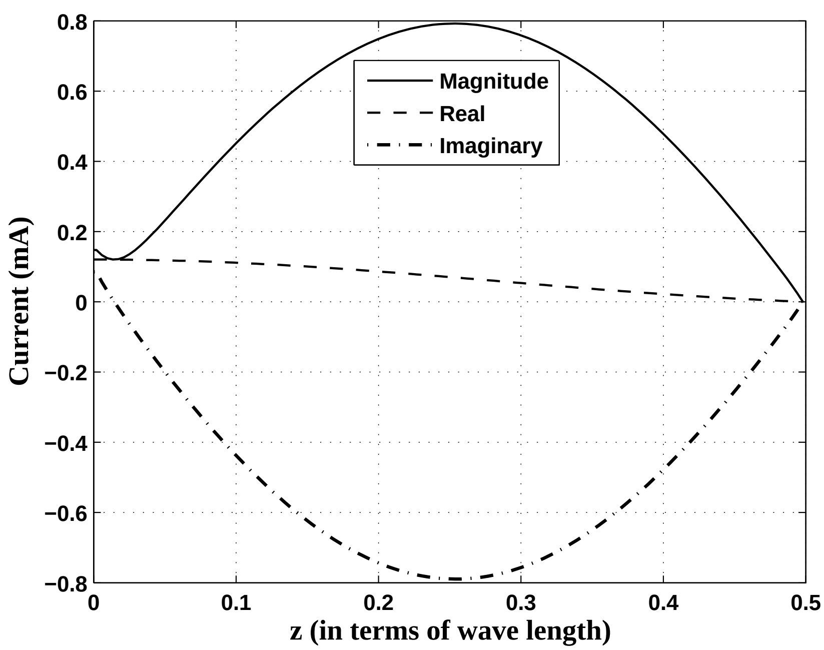

Figure 3: Current distribution for $2l = \lambda$ and $a = 0.00001l$.

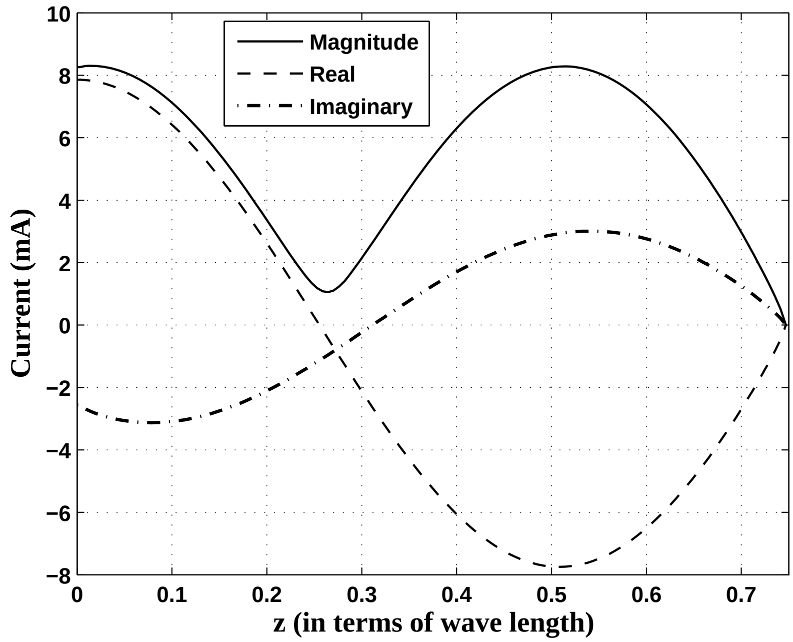

Figure 4: Current distribution for $2l = 1.5\lambda$ and $a = 0.001l$.

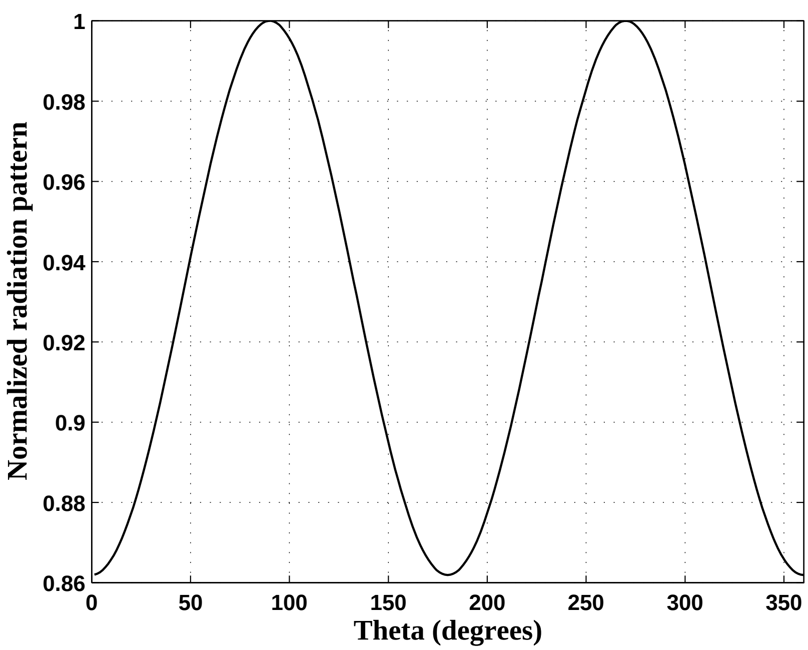

Figure 5: Normalized radiation pattern for $2l = 0.4\lambda$ and $a = 0.0001l$.

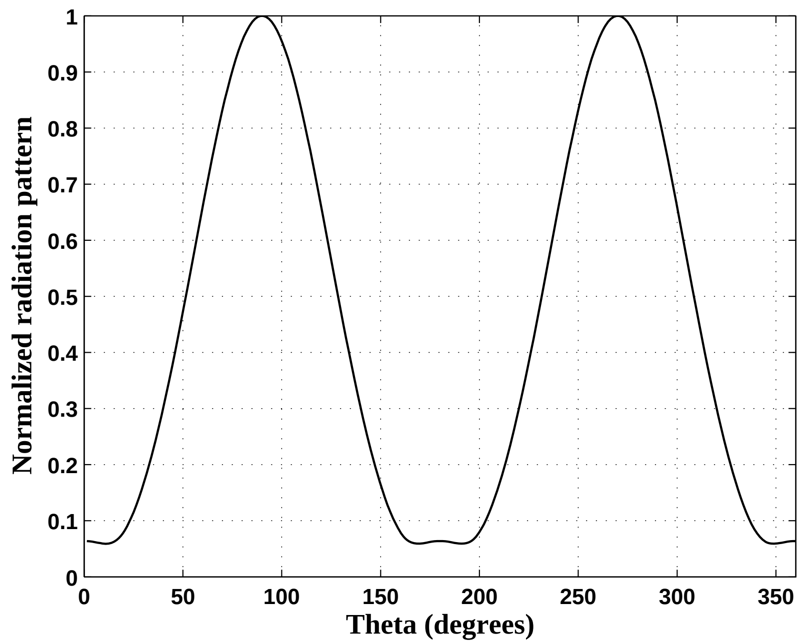

Figure 6: Normalized radiation pattern for $2l = \lambda$ and $a = 0.00001l$.

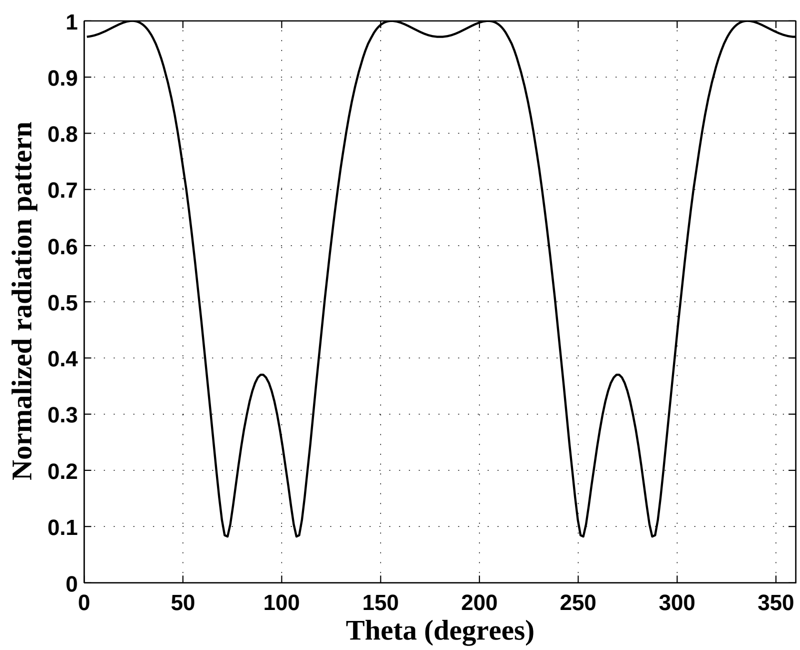

Figure 7: Normalized radiation pattern for $2l = 1.5\lambda$ and $a = 0.001l$.

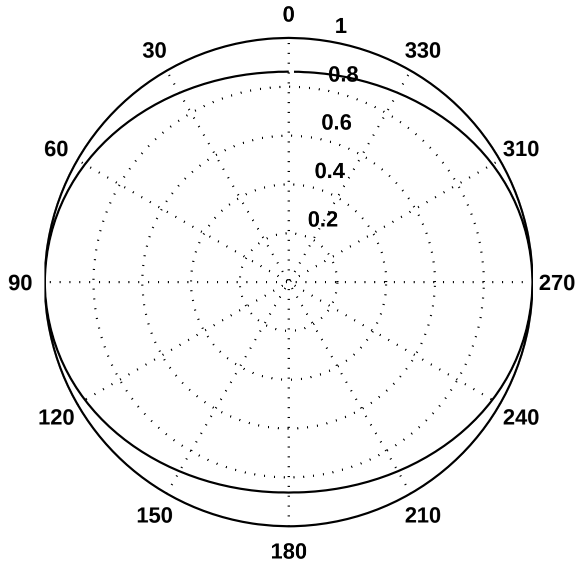

Figure 8: Normalized radiation pattern in polar system for $2l = 0.4\lambda$ and $a = 0.0001l$.

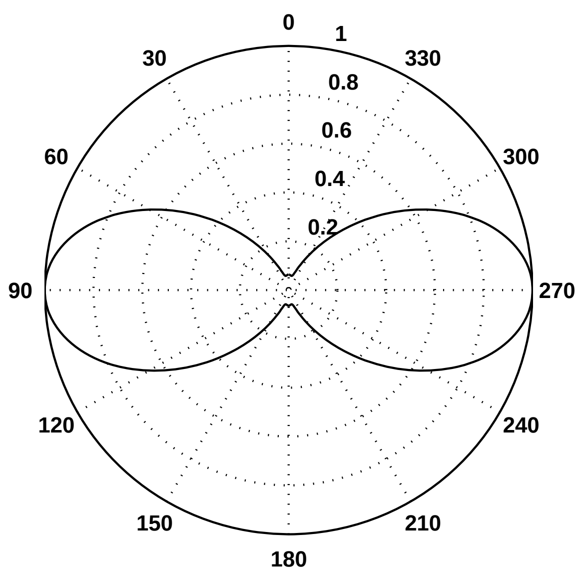

Figure 9: Normalized radiation pattern in polar system for $2l = \lambda$ and $a = 0.00001l$.

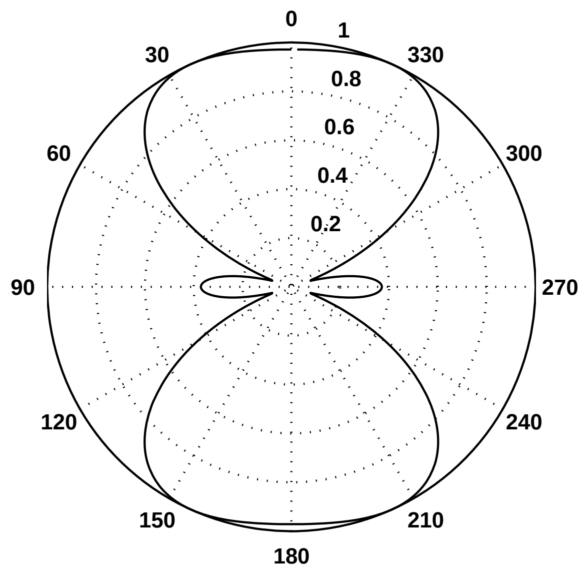

Figure 10: Normalized radiation pattern in polar system for $2l = 1.5\lambda$ and $a = 0.001l$.

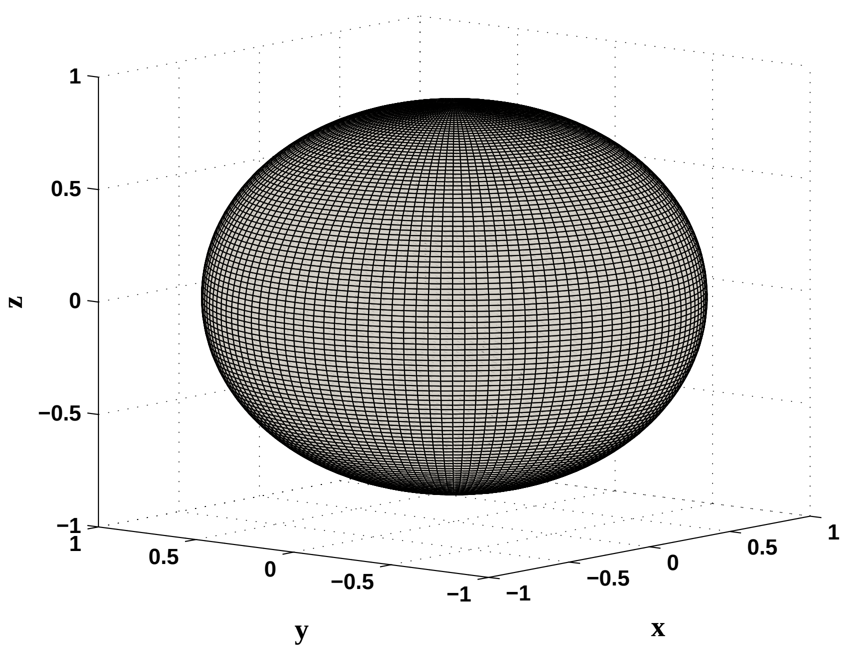

Figure 11: Three-dimensional radiation pattern for $2l = 0.4\lambda$ and $a = 0.0001l$.

ISSN:2456-4834
Available online at journals.aijr.org

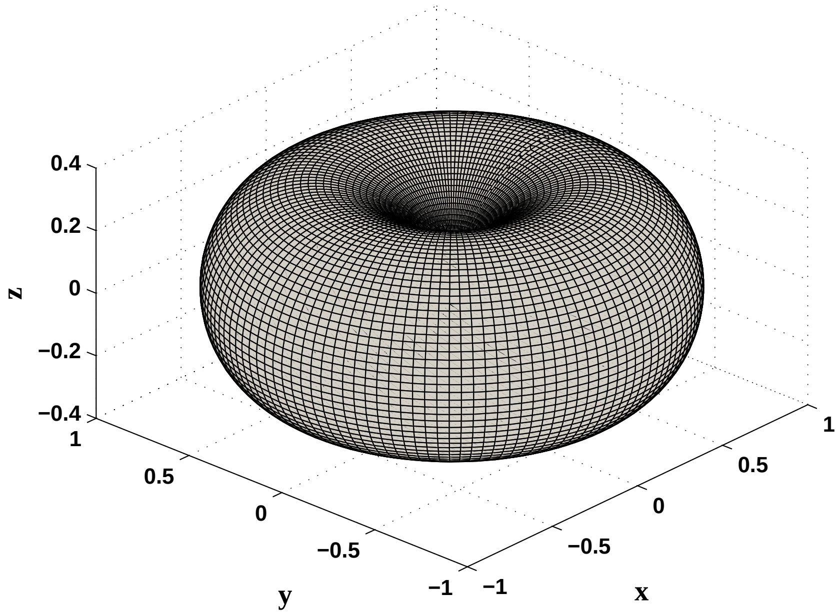

Figure 12: Three-dimensional radiation pattern for $2l = \lambda$ and $a = 0.00001l$.

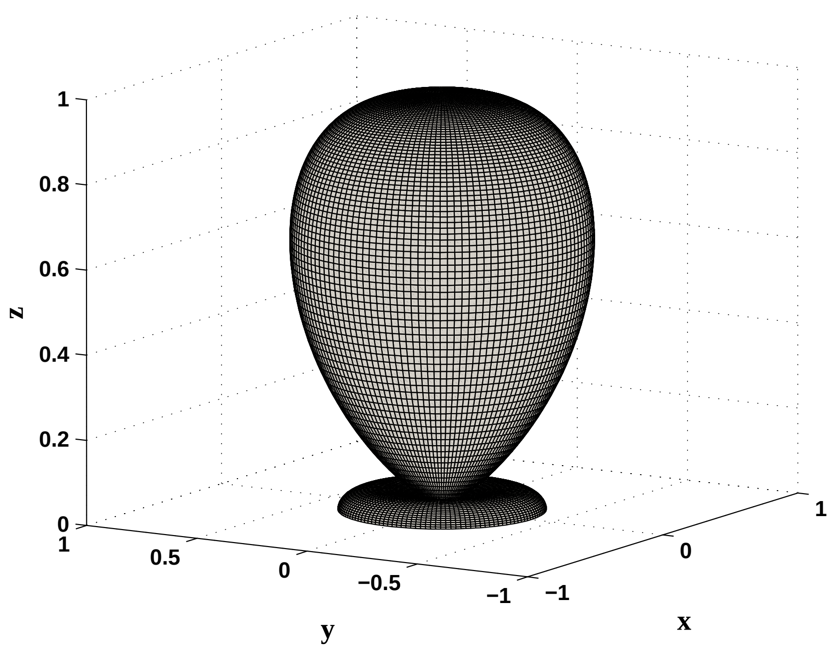

Figure 13: Three-dimensional radiation pattern for $2l = 1.5\lambda$, $a = 0.001l$ and $z > 0$.

5 Conclusion

The presented method in this paper was applied to solve Hallén's integral equation for a dipole antenna of perfectly conducting material using a special representation of triangular functions. This method reduces the Hallén's integral equation to a linear system of algebraic equations. The problem was described in detail, and illustrative computations were given for current distributions and radiation patterns. The advantage of the proposed method is its flexibility to be generalized for applying in analysis of other radiating structures. Moreover, the numerical results confirm its computational efficiency as an integral equation approach. At last, a similar formulation can be applied in solution of Pocklington's integral equation for analysis of radiating bodies of perfectly conducting material.

6 Declarations

6.1 Competing Interests

There is no conflict of interest in this work.

6.2 Publisher's Note

AIJR remains neutral with regard to jurisdictional claims in institutional affiliations.

How to Cite this Article:

S. Hatamzadeh and Z. Masouri, "Applying a Set of Orthogonal Basis Functions in Numerical Solution of Hallen's Integral Equation for Dipole Antenna of Perfectly Conducting Material",*J. Mod. Mater.*, vol. 9, no. 1, pp. 36–49, Nov. 2022. https://doi.org/10.21467/jmm.9.1.36-49

References

[1] H. Chai, J. Lv, and Y. Bao, "Numerical solutions of hypersingular integral equations for stress intensity factors of planar embedded interface cracks and their correlations with bimaterial parameters," *International Journal of Solids and Structures*, Vol. 202, pp. 184–194, 2020.

[2] M.I. Azis, M. Abbaszadeh, M. Dehghan, and I. Solekhudin, "A boundary-only integral equation method for parabolic problems of another class of anisotropic functionally graded materials," *Materials Today Communications*, Vol. 26, Article 101956, 2021.

[3] K.B. Hamzah, N.M.A. Nik Long, N. Senu, and Z.K. Eshkuvatov, "Numerical solution for the thermally insulated cracks in bonded dissimilar materials using hypersingular integral equations," *Applied Mathematical Modelling*, Vol. 91, pp. 358–373, 2021.

[4] T.L. Yookesh, M. Chithambarathanu, E. Boopathi Kumar, and P. Sudam Sekhar, "Efficiency of iterative filtering method for solving Volterra fuzzy integral equations with a delay and material investigation," *Materials Today Proceedings*, Vol. 47, Part 17, pp. 6101–6104, 2021.

[5] K. Ranjith, "Spectral formulation of the boundary integral equation method for antiplane problems," *Mechanics of Materials*, Vol. 165, Article 104177, 2022.

[6] Z.Y. Ai, Y.H. Jiang, and Z.H. Chu, "Dual integral equation solution of eccentricly loaded rectangular rigid foundation embedded in layered transversely isotropic soils," *Computers and Geotechnics*, Vol. 126, Article 103755, 2020.

[7] R.S. Elliot, *Antenna Theory and Design*, Englewood Cliffs, New Jersey, Prentice Hall, 1981.

[8] L.M. Delves and J.L. Mohamed, *Computational Methods for Integral Equations*, Cambridge University Press, Cambridge, 1985.

[9] A. Deb, A. Dasgupta, and G. Sarkar, "A new set of orthogonal functions and its application to the analysis of dynamic systems," *Journal of the Franklin Institute*, Vol. 343, 1–26, 2006.

[10] C.A. Balanis, *Antenna Theory: Analysis and Design*, Wiley, New York, 1982.

[11] W.L. Stutsman and G.A. Thiele, *Antenna Theory and Design*, John Wiley & Sons, 1998.