**RESEARCH ARTICLE**

# Forward Time Centered Space Scheme for the Solution of Transport Equation

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## ABSTRACT

Transport phenomenon is central for understanding many procedures in numerous sciences, transport phenomena can be described by the partial differential equation. Forward Time Centered Space scheme give ample numerical solutions of the transport equation. In this paper, we apply Forward Time Centered Space scheme to solve a non-trivial transport problem using different step sizes of time (t) and space (x). We use MATLAB software to get the numerical results. The numerical simulation presents that the FTCS scheme is more stable and closer to the exact solution when we decrease step sizes of *t, x and* α more and more.

**Keywords:** Transport equation; Forward Time Centered Space scheme; numerical simulations.

## 1 Introduction

The transport phenomenon is a central for understanding many procedures in numerous sciences as engineering, agriculture, meteorology, physiology, biology, analytical chemistry, materials science, pharmacy, and other areas. Transport phenomenon is a well-developed and very useful branch of physics that used in many areas of applied science [1]. Transport equation describes many transport phenomena such as heat transfer, mass transfer, fluid dynamics, spread of neutrons, sound propagation, and other applications of physics [2]. Differential equations have been applied to describe many natural phenomena in many areas [3] - [8]. Transport equation that has different forms is a general partial differential equation. One dimensional transport equation that we use in this paper is:

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} - \alpha\frac{\partial^2 T}{\partial x^2} = 0 \qquad (1)$$

where T is a passive scalar (e.g. temperature concentration) which is being transported with a known velocity u(x, t). If T is the temperature, then α is called the thermal diffusivity coefficient. The development in computer hardware and numerical methods algorithms has led to use numerical methods in solving many applied problems. Therefore, the recent years have witnessed shifting from analytical techniques to computer-oriented numerical methods. The primary appeal of numerical methods is that they provide solutions for many problems, which are not amenable through analytical solution. The development in modern digital computers paved the way for the development of efficient finite difference

methods, which are employed for solving particularly partial differential equations [9]. Finite difference methods present sufficient numerical results in a simpler and more efficient way. Finite difference methods are used for solving partial differential equation which gives the required solutions while analytical methods are often fail [10]. In this paper, we apply FTCS (Forward Time Centered Space) scheme that is an example of explicit finite difference scheme to solve a non-trivial transport problem which has sharp continuous initial condition.

## 2    Forward Time Centred Space Scheme

For using FTCS scheme to solve the equation (1) we take the forward time and central space at point $(x_j, t_n)$, and apply them on the equation (1) [11]. So we obtain:

$$\frac{T_j^{n+1} - T_j^n}{\Delta t} + \frac{u(T_{j+1}^n - T_{j-1}^n)}{2\Delta x} - \frac{\alpha(T_{j-1}^n - 2T_j^n + T_{j+1}^n)}{\Delta x^2} = 0 \quad (2)$$

The equation (2) can be written as an algorithm:

$$T_j^{n+1} = (s + 0.5C)T_{j-1}^n + (1 - 2s)T_j^n + (s - 0.5C)T_{j+1}^n \quad (3)$$

Where $s = \dfrac{\alpha \Delta t}{\Delta x^2}$ and $C = \dfrac{u \Delta t}{\Delta x}$.

We can write equation (3) for proposal of programming as follows:

$$T_j^n = (s + 0.5C)T_{j-1}^{n-1} + (1 - 2s)T_j^{n-1} + (s - 0.5C)T_{j+1}^{n-1} \quad (4)$$

According to the Von Neuman stability analysis; the stability condition for FTCS scheme is as follows [12]:

$$0 \le C^2 \le 2s \le 1. \quad (5)$$

The local truncation error for FTCS scheme is [11]:

$$Cu(\frac{\Delta x}{2})\frac{\partial^2 T}{\partial x^2} - [C\alpha\Delta x - u(\frac{\Delta x^2}{6})(1 + 2C^2)]\frac{\partial^3 T}{\partial x^3} \quad (6)$$

## 3    Analytical Solution

The separation of variables technique is used to get the exact solution for equation (1), it was obtained by Fletcher [11], as follows:

$$T(x,t) = 0.5 - \frac{2}{\pi}\sum_{k=1}^{N}\sin\left[(2k-1)\frac{\pi(x - ut)}{L}\right]\frac{\exp[-\alpha(2k-1)^2\pi^2 t / L^2]}{2k - 1} \quad (7)$$

We indicate that the initial and boundary conditions are:

T(x,0)=1.0 if $-2 \le x < 0$ (initial condition) (8)
T(x,0)=0.0 if $0 \le x \le 2$   (initial condition) (9)
T(-2,t)=1.0 if $t > 0$ (boundary condition)  (10)
T(2,t)=0.0 if $t > 0$ (boundary condition)    (11)

## 4    Numerical Simulation Results

Numerical simulations are carried out using MATLAB program to obtain FTCS scheme results for transport equation at specific values of space (x) that are {-2.0, -1.5, -1.0, -0.5, 0.0, 0.5, 1.0, 1.5, 2.0} when the value of time at $t = 1.0$, for different values of $\Delta x$, $\Delta t$ and $\alpha$.

We use a FTCS scheme formula (3), the exact solution formula (7) and the initial and boundary conditions to compare between FTCS scheme and the exact solution results.

### 4.1    FTCS Scheme Results (when $\Delta x = 0.5$, $\Delta t = 0.1$ and $\alpha = 1.0$)

To run the MATLAB program, we assume in the first case the following values ($\Delta x = 0.5$, $\Delta t = 0.1$, $u = 1.0$, $\alpha = 1.0$ and $L = 4$), to obtain the results of FTCS scheme for transport equation and exact solution for transport equation. We pick $\Delta x = 0.5$ to generate the all values of x that are {-2.0, -1.5, -1.0, -0.5, 0.0, 0.5, 1.0, 1.5, 2.0}, we select $\Delta t = 0.1$ to get the value of time when $t = 1.0$, $L = 4$ represents the length of axis x $(-2 \le x \le 2)$ which represents the boundary for the problem, and we select $u = 1.0$ and $\alpha = 1.0$ to satisfy the stability condition in addition to the values of $\Delta x$ and $\Delta t$ We consider these values as a first case to be:-

$$s = 0.4 \text{ where } s = \frac{\alpha \Delta t}{\Delta x^2}$$

and $C = 0.2$ where $C = \dfrac{u \Delta t}{\Delta x}$.

We select these values that they satisfy the condition (5). We use specific values of space (x) that are {-2.0, -1.5, -1.0, -0.5, 0.0, 0.5, 1.0, 1.5, 2.0}

when the value of time at $t = 1.0$ to get the following table:

**Table 1:** *Results of FTCS scheme with Exact solution for transport equation when* $\Delta x = 0.5$, $\Delta t = 0.1$ *and* $\alpha = 1.0$

| Value of x | FTCS Scheme | Exact Solution | Absolute Error |
|---|---|---|---|
| -2.0 | 1.0000 | 1.0000 | 0.0000 |
| -1.5 | 0.9625 | 0.8171 | 0.1454 |
| -1.0 | 0.9045 | 0.8427 | 0.0618 |
| -0.5 | 0.8213 | 0.8171 | 0.0042 |
| 0.0 | 0.7104 | 0.7435 | 0.0331 |
| 0.5 | 0.5759 | 0.6322 | 0.0563 |
| 1.0 | 0.4205 | 0.5000 | 0.0795 |
| 1.5 | 0.2410 | 0.3678 | 0.1268 |
| 2.0 | 0.0000 | 0.0000 | 0.0000 |

We notice from the Table 1 that the sum of absolute error is 0.5071 and the average of absolute error is 0.05634.
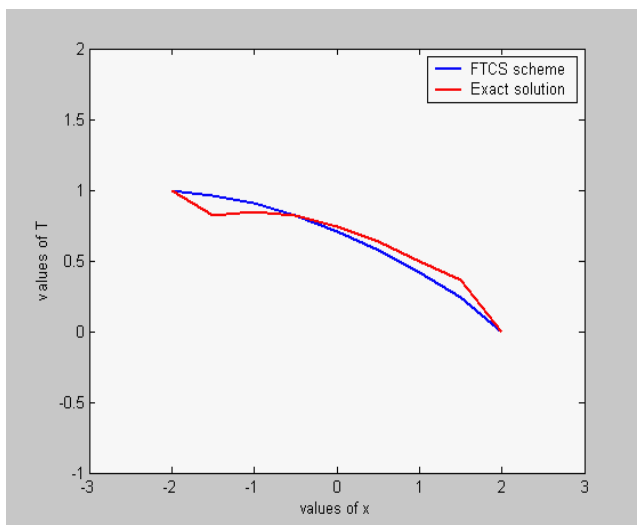


**Figure 1:** *FTCS scheme and Exact solution of transport equation together when* $\Delta x = 0.5$, $\Delta t = 0.1$ *and* $\alpha = 1.0$

Figure 1 describes the results of FTCS scheme with exact solution together for transport equation at specific values of space (x) which are {-2.0, -1.5, -1.0, -0.5, 0.0, 0.5, 1.0, 1.5, 2.0} and the value of time at $t = 1.0$ when $\Delta x = 0.5$ $\Delta t = 0.1$ and $\alpha = 1.0$

### 4.2 FTCS Scheme Results (when $\Delta x = 0.25$, $\Delta t = 0.05$ and $\alpha = 0.5$)

To run the MATLAB program in the second case, we suppose the following values ( $\Delta x = 0.25$, $\Delta t = 0.05$, $u = 1.0$, $\alpha = 0.5$ and $L = 4$ ), to obtain the results in this case of FTCS scheme for transport equation and exact solution for transport equation. We pick $\Delta x = 0.25$ to generate the all values of x that are {-2.0, -1.5, -1.0, -0.5, 0.0, 0.5, 1.0, 1.5, 2.0}, we select $\Delta t = 0.05$ to get the value of time when $t = 1.0$, $L = 4$ represents the length of axis x ( $-2 \leq x \leq 2$) which represents the boundary for the problem, and we select $u = 1.0$ and $\alpha = 0.5$ to satisfy the stability condition (5) in addition to the values of $\Delta x$ and $\Delta t$.

We consider these values as a second case to be-

$s = 0.4$ Where s $= \dfrac{\alpha \Delta t}{\Delta x^2}$

and $C = 0.2$ where C $= \dfrac{u \Delta t}{\Delta x}$.

We select these values so that they satisfy the condition (5). We use specific values of space (x) which are {-2.0, -1.5, -1.0, -0.5, 0.0, 0.5, 1.0, 1.5, 2.0} when $t = 1.0$ to get the following table according to second case:

Table 2: Results of FTCS scheme with exact solution of transport equation when $\Delta x = 0.25$, $\Delta t = 0.05$ and $\alpha = 0.5$

| Value of X | FTCS Scheme | Exact Solution | Absolute Error |
|---|---|---|---|
| -2.0 | 1.0000 | 1.0000 | 0.0000 |
| -1.5 | 0.9926 | 0.9270 | 0.0656 |
| -1.0 | 0.9712 | 0.9545 | 0.0167 |
| -0.5 | 0.9186 | 0.9270 | 0.0084 |
| 0.0 | 0.8149 | 0.8400 | 0.0251 |
| 0.5 | 0.6533 | 0.6912 | 0.0379 |
| 1.0 | 0.4537 | 0.5000 | 0.0463 |
| 1.5 | 0.2491 | 0.3088 | 0.0597 |
| 2.0 | 0.0000 | 0.0000 | 0.0000 |

We notice from the Table 2 that the sum of absolute error is 0.2597 and the average of

absolute error is 0.02886. We notice that the results of FTCS scheme for transport equation in the second case are more accurate than the first case because $\Delta x = 0.25$, $\Delta t = 0.05$ and $\alpha = 0.5$ is used in the second case while $\Delta x = 0.5$, $\Delta t = 0.1$ and $\alpha = 1.0$ is used in the first case. This means the results of FTCS scheme for transport equation in the second case is more improve than the results of FTCS scheme for transport equation in the first case.
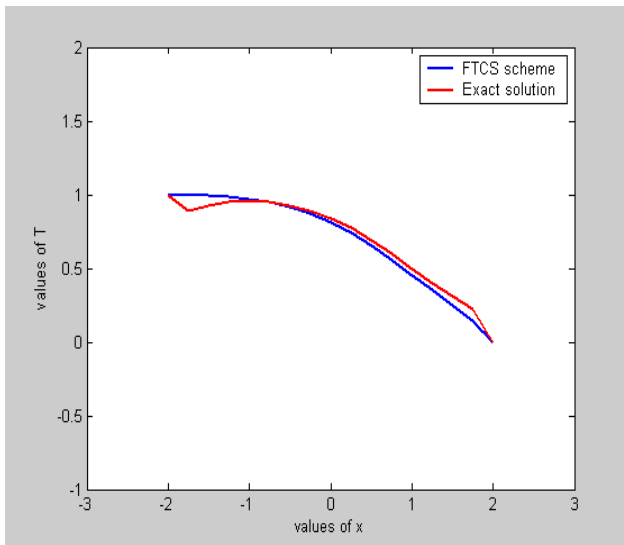


**Figure 2:** *FTCS scheme and Exact solution together of transport equation when $\Delta x = 0.25$, $\Delta t = 0.05$ and $\alpha = 0.5$*

The Figure 2 describes the results of FTCS scheme with exact solution together for transport equation at specific values of space (x) which are {-2.0, -1.5, -1.0, -0.5, 0.0, 0.5, 1.0, 1.5, 2.0} and the value of time at $t = 1.0$ when $\Delta x = 0.25$ $\Delta t = 0.05$ and $\alpha = 0.5$.

**4.3 FTCS Scheme Results** (when $\Delta x = 0.05$, $\Delta t = 0.01$ and $\alpha = 0.0625$)

To run the MATLAB program in third case, we suppose the following values ($\Delta x = 0.05$, $\Delta t = 0.01$, $u = 1.0$, $\alpha = 0.0625$ and $L = 4$), to obtain the results in this case of FTCS scheme for transport equation and exact solution of transport equation. We pick $\Delta x = 0.25$ to generate the all values of x that are {-2.0, -1.5, -1.0, -0.5, 0.0, 0.5, 1.0, 1.5, 2.0}, we select $\Delta t = 0.05$ to get the value of time when

$t = 1.0$, we choose $L = 4$ represents the length of axis x $(-2 \leq x \leq 2)$ which represents the boundary for the problem, and we select $u = 1.0$ and $\alpha = 0.5$ to satisfy the stability condition (5) in addition to the values of $\Delta x$ and $\Delta t$.

We consider these values as a third case to be:-

$$s = 0.25 \text{ Where } s = \frac{\alpha \, \Delta t}{\Delta x^2}$$

and $C = 0.2$ where $C = \frac{u \Delta t}{\Delta x}$.

We select these values to satisfy the condition (5). We also use specific values of space (x) that are {-2.0, -1.5, -1.0, -0.5, 0.0, 0.5, 1.0, 1.5, 2.0} when $t = 1.0$ to get the following third case table:

**Table 3:** *Results of FTCS scheme with Exact solution of transport equation when $\Delta x = 0.05$, $\Delta t = 0.01$ and $\alpha = 0.0625$*

| Value of x | FTCS Scheme | Exact Solution | Absolute Error |
|---|---|---|---|
| -2.0 | 1.0000 | 1.0000 | 0.0000 |
| -1.5 | 1.0000 | 1.0000 | 0.0000 |
| -1.0 | 1.0000 | 1.0000 | 0.0000 |
| -0.5 | 1.0000 | 1.0000 | 0.0000 |
| 0.0 | 0.9966 | 0.9977 | 0.0011 |
| 0.5 | 0.8946 | 0.9214 | 0.0268 |
| 1.0 | 0.4142 | 0.5000 | 0.0858 |
| 1.5 | 0.0440 | 0.0786 | 0.0346 |
| 2.0 | 0.0000 | 0.0000 | 0.0000 |

We notice from the Table 3 that the sum of absolute error is 0.1483 and the average of absolute error is 0.016478. We notice that the results of FTCS scheme for transport equation in the third case are more accurate than those of the first case and the second case because $\Delta x = 0.05$, $\Delta t = 0.01$ and $\alpha = 0.0625$ are used in the third case, while $\Delta x = 0.25$, $\Delta t = 0.05$ and $\alpha = 0.5$ are used in the second case and in the first case $\Delta x = 0.5$, $\Delta t = 0.1$ and $\alpha = 1.0$ are used. This means the results of FTCS scheme for transport equation in the third case are the most improve the results of FTCS scheme for transport equation in the first case, and in the second case. It is noticed that whenever $\Delta x$, $\Delta t$ and $\alpha$

decrease, the sum and the average of absolute error in FTCS scheme decrease, this means that the results become more accurate and close to the exact solution whenever $\Delta x$, $\Delta t$ and $\alpha$ are small.
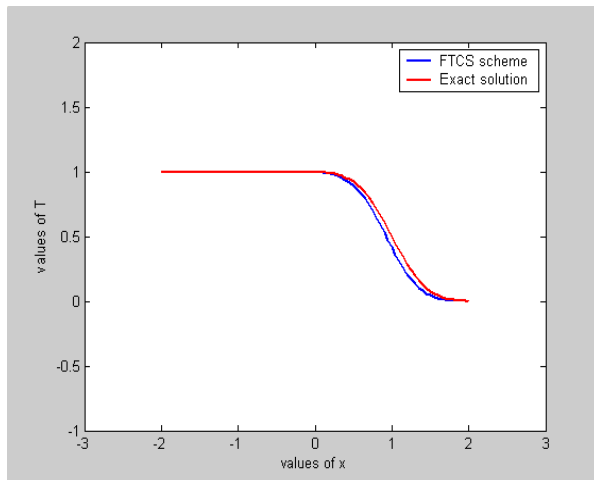


**Figure 3:** *FTCS scheme and Exact solution together of transport equation when* $\Delta x = 0.05$, $\Delta t = 0.01$ *and* $\alpha = 0.0625$

The Figure 3 describes the results of FTCS scheme with exact solution for transport equation at specific values of space (x) that are {-2.0, -1.5, -1.0, -0.5, 0.0, 0.5, 1.0, 1.5, 2.0} and the value of time at $t = 1.0$ when $\Delta x = 0.05$, $\Delta t = 0.01$ and $\alpha = 0.0625$.

## 5    Conclusions

Transport equation has been considered for describing transport phenomena in many areas, that is partial differential equation. We have applied Forward Time Centred Space scheme to solve a non-trivial transport problem using different step sizes of time (t) and space (x). MATLAB software has been used to get the results. The numerical simulations presents three different values of $\Delta x$, $\Delta t$ and $\alpha$ with fixing other parameter values at specific values of space (x) that are {-2.0, -1.5, -1.0, -0.5, 0.0, 0.5, 1.0, 1.5, 2.0} when the value of time at $t = 1.0$. the results show FTCS scheme is more stable and closer to the exact solution through the absolute error when we decrease step sizes of t, x and $\alpha$ more and more.

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