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Quantum Control Modelling, Methods, and Applications

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ABSTRACT

This review concerns quantum control results and methods that, over the years, have been used in the various operations involving quantum systems. Most of these methods have been originally developed outside the context of quantum physics, and, then, adapted to take into account the specificities of the various quantum physical platforms. Quantum control consists in designing adequate control signals required to efficiently manipulate systems conforming the laws of quantum mechanics in order to ensure the associated desired behaviours and performances. This work attempts to provide a thorough and self-contained introduction and review of the various quantum control theories and their applications. It encompasses issues spanning quantum control modelling, problem formulation, concepts of controllability, as well as a selection of the main control theories. Given the vastness of the field, we tried our best to be as concise as possible, and, for the details, the reader is pointed out to a profusion of references. The contents of the review are organized in the three major classes of control problems - open-loop control, closed-loop learning control, and feedback control - and, for each one of them, we present the main developments in quantum control theory. Finally, concerning the importance of attaining robustness and reliability due to inherent fragility of quantum systems, methods for quantum robust control are also surveyed.

Keywords: Quantum Systems, Quantum Control Modelling, Quantum Control Methods.

1. Introduction

Quantum control deals with active manipulation of chemical and physical actions in the atomic and molecular scale. Recent advances of quantum technologies enable practical applications of quantum control in different scientific areas such as quantum optics, [1], cavity quantum electro-dynamics (QED), [2], atomic spin ensembles, [3], ion trapping, [4], Bose–Einstein condensation, [5], nitrogen vacancy centres, [6], amongst others, which reveals the high importance of manipulating quantum phenomena. Quantum phenomena, such as entanglement and coherence, both rooted in the superposition principle, are at the heart of disparity between classical and quantum physics. Entanglement is a phenomenon in which quantum entities are created and/or manipulated such that none of them can be described without referencing the others, [7]. Quantum coherence depicts the idea of a superpositioning, that is a qubit is in a superposition of the “0” and “1” states, which results in a speed-up over a variety of classical algorithms. When such a state experiences decoherence, however, all of its quantumness is typically lost and the advantage vanishes, [8]. From quantum control perspective, the main aim of controlling a system is the manipulation of quantum state trajectory and state evolution via an external control combined with quantum system properties to achieve the desired state. This paper provides a thorough and self-contained introduction and review of the various quantum control theories and their applications. In the sequel, we address some of the main issues related to quantum control systems.

Controllability: One of the basic issues that one need to consider for a quantum systems under study concerns to its controllability, which relates to the possibility of driving a quantum system to a target state by a proper control signal. The practical importance of controllability is due to the close connection between the universality of quantum computing and the possibility of attaining small scale transformations, [9, 10]. For controllable quantum systems, the challenging task of quantum control involves the development of control strategies in addition to effectively manipulate multi qubit systems while tackling with the decoherence caused by the unwanted environmental interactions and other forms of noise to fulfill the quantum control objectives, [11].

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Open-loop Control: Open-loop control has been amongst the early paradigms of quantum control methods. In this method, the control law is determined and designed prior to an experiment and is later used during the experiment without any modifications. Coherent control strategy, [12], has been widely used in order to ensure quantum coherence in controlled systems, having accomplished great success in quantum control of chemical reactions, [13]. Open-loop coherent control is an early work on quantum control, in which an open-loop coherent controller acts by applying time dependent potential to the system. Optimal control theory has also been used for designing open-loop coherent control schemes, aiming to achieve the best ways of fulfilling quantum control tasks. It formulates the quantum control problem as looking for a set of admissible controls that satisfy the system dynamics while minimizing a cost functional in order to obtain a control law, [14, 15, 16]. The cost functional is chosen according to the properties of quantum system, being, as notable examples, time optimal, [17], energy optimal, [15], and optimal unitary transformation, [18], or a combination of these. There are already various methods to design the optimal controller in quantum systems, which varies in cost functional, Hamiltonian function, and solving process, [19]. The investigation of optimal controls to quantum systems has been done through adaptation of conventional optimal control tools such as convergent iterative algorithms, and Pontryagin maximum principle, [14]. Quantum optimal control has been exploited for the improvement of process performance in various areas of quantum technologies, e.g., looking for the optimum sensitivity in magnetic resonance experiments, [20].

Bang-bang control and geometric control are two other common control methods used in quantum systems. Bang-bang control theory is a type of simple switch control, providing an interesting case for bounded control. For instance, bounded bang-bang control method has been utilized to obtain optimal time for population transfer problem. This strategy is used to suppress decoherence and manipulate a class of states, [21]. The other strategy, which is geometric control, has been shown to be satisfactory for low-level systems combined with the Bloch Sphere. Moreover, optimal control theory and geometric control have been used in the quantum state transformation as in [22]. It is worth mentioning that closed-loop optimal control strategies have been used to manipulate quantum entanglement, [23], identifying the parameters of system Hamiltonian, [24], and also to track quantum states, [25]. Despite being applicable to wide class of optimization problems due to the large problem formulation flexibility, the main weakness of optimal control is due to the two-point boundary value problem, that, typically, are hard to solve, and the computational performance of the associated iterative procedures depend strongly on their initialization. Therefore, it is necessary to guess an initial control function to initialize the process and, then, improve it by successive iterations. This procedure demands complex computations, and its application in quantum physics is limited since a quick response is required, [19].

Control operations of coherent control are unitary transformations, so it may not be possible or very difficult to control some quantum systems directly through coherent resources. In open-loop control, for such quantum systems with poor controllability, like finite dimensional open quantum systems with Markovian dynamics, [26], additional strategies or resources such as incoherent control, [27], can be established to enhance the performance of quantum control systems. In fact, incoherent control strategies have been identified as an open-loop technique to assist coherent quantum control, [28]. Although quantum measurements are not suitable to achieve coherent control objectives due to the fact that they may destroy coherent characteristics of quantum system, they can be utilized as a practical control tool to improve controllability. Overall, despite the success of open-loop control strategies to control some simple quantum systems, their applications have significant drawbacks in dealing with perturbations and suppressing decoherence for complex quantum systems.

Closed-loop Learning Control: Learning control implies a closed-loop iterative procedure in which every cycle of the loop is performed with an updated sample, [11]. To do so, several learning algorithms have been employed in which the stability in the presence of disturbances is highly important. Typical optimization methods in quantum control include gradient-based methods, for instance D-MORPH search algorithm, [29], stochastic searching methods, such as Genetic Algorithm (GA), [30], Differential Evolution (DE), [31], Particle Swarm Optimization (PSO), [32], and Reinforcement Learning (RL) Methods, [33, 34], which are global search methodologies that may be combined with the direction of gradient-like rewards. Stochastic learning control methods have also been considered in the context of experimental adaptive feedback control, [35]. In adaptive feedback control, a loop is closed, and the results of measurements are used for the evaluation of success of the applied control and in order to refine it, until the best approximation to the control target is obtained, [36]. A comparison between gradient decent method, GA, and DE methods has been done in [37].

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Feedback Control: Quantum feedback control is another class of methods for preparation and manipulation of quantum systems. We start with Lyapunov control as an open-loop quantum control strategy with feedback design. The feedback design is used for the construction of control fields which are later applied to the quantum systems in an open-loop way, that is, the output state of the designed feedback control system is obtained from a mathematical model. Since the feedback information acquisition via measurement demolishes the quantum state, it would be difficult to straightly apply Lyapunov approaches to quantum feedback controller design. Therefore, the feedback control design would be done via simulation, and a sequence of controls would be obtained and applied to the quantum system to be controlled in an open-loop way. The procedure in Lyapunov control design include three main parts including the selection of the Lyapunov function, designing the feedback law, and analysing asymptotic convergence, [38, 39]. In Lyapunov control design method, the control law is obtained from the solution of the partial differential equation via Lyapunov indirect stability theorem with no iterations, which subsequently results in rapid quantum control. The method has been successfully used for driving quantum systems into decoherence-free subspaces, controlling open quantum systems in decoherence-free subspaces, controlling entanglement generation for distant atoms, and especially for quantum state transfer. Another closed-loop method, helping attain improved properties for quantum systems, is variable structure control, [40]. This method provides the possibility of altering the structure of controller conforming to the defined switching logic. Therefore, the structure of the control law differs according to the position of the state trajectory; the method provides fast switching from one smooth control law to another, and is appropriate for qubit state preparation, [41, 42]. Generally, in feedback control method, the system's quantum state or trajectory is used to evolve the system towards some desired outcome, making quantum control more effective. We will study the rest of common methods in quantum feedback control in two main typical categories: measurement-based feedback control, and coherent feedback control. In measurement-based method, some observables or signals of the system are measured, and the result is processed in a controller in order construct a classical control signal for driving a proper actuator that exerts direct influence on the quantum system to be controlled. Although the system under control is a quantum system, the controller may not be a quantum one; classical or hybrid quantum-classical controllers may also be used. Two fundamental techniques for information accession are projective measurement (or von Neumann measurement), [43], and continuous weak measurement [44]. Markovian quantum feedback, [45], is one of the main paradigms of quantum feedback control based on measurement, with applications in several physical problems. This method ignores any time delay and the designed controller is memoryless. Another important paradigm is Bayesian feedback method, in which the dynamical equation is non-Markovian, and its process contains two stages. The first stage is to estimate the dynamical variables. The best estimations can be obtained continuously from the measurement records. The second stage is the feedback control, in which the estimates are fed back to the system dynamics controller. This method has already been applied in quantum state preparation, [46], and also for quantum error correction, [47]. Weak and non-demolition measurements make it possible to change the evolution continuously via Hamiltonian feedback that depends on the measurement record, [48], but for most problems, it is not practical to design quantum non-demolition measurements. Weak measurement can be modelled via a stochastic master equation by introducing an auxiliary system that is weakly coupled to the system under study. The applications of this method include weak control of entanglement generation, [49], and control of a single qubit, [50]. The main drawbacks of measurement-based feedback are: perturbation of the quantum system through measurement, limited information processing speed, and the fact that the information taken from measurement is stochastic. Coherent-feedback control is another method used in quantum feedback control. In this method, a feedback controller, which is a full quantum system, is connected to the quantum system under control in a feedback loop configuration that preserves quantum coherence. Coherent control with quantum (coherent) feedback is especially useful for the generation, transformation, and transfer of entanglement that cannot be achieved by coherent control with classical feedback. The two noise-reducing controllers, H^{∞} controller, [51], and linear quadratic Gaussian controllers, [52], have natural coherence control analogues. By optimizing a quadratic cost functional, quantum linear-quadratic-Gaussian control can be developed in order to seek an optimal feedback control law for a stochastic linear system, [53].

Robust Control: Robustness has been an essential property for quantum technologies due to inevitable disturbances and uncertainties. H^{∞} controller synthesis problem can be formulated for a class of non-commutative linear stochastic systems including several interesting examples in quantum world, [51]. Sliding mode control is another method, [54], yielding control robustness. Sliding mode control can be considered as one of the main approaches in variable structure control of classical systems, however, it can be used to deal with uncertainties in the quantum

feedback control. Sliding mode control is also suitable for quantum systems evolving with nonlinear equations, [55], quantum systems with bounded uncertainties in their system Hamiltonian, [54], two-level quantum systems, [56], and also for robust control of a single qubit, [57]. In linear quadratic Gaussian design, referred to as risk neutral optimal control, cost is additive while in linear exponential quadratic Gaussian control, known as risk sensitive optimal control, cost is multiplicative. It is known that risk sensitive controllers have a close connection with robustness. Furthermore, various robustness properties of quantum filtering and estimation have been presented, e,g., guaranteed-cost, [53], and risk-sensitive filtering, [58]. Quantum ensemble control, [59], is another robust control method, which involves controlling a continuum of dynamical systems with different values of parameters characterizing the system dynamics by using the same control signal. In addition, sequential convex programming, [60], is a local optimization method that can be used to ensure error suppression. Dynamical decoupling, [61], can also be mentioned as another favourable method for robust control of quantum systems. This method is consistent with other desired functionalities, e.g., quantum gates, [62].

The remainder of this work is organized as follows: We first review several approaches for modelling a quantum system to be controlled. Then, we turn to the context of controllability of quantum systems and present some of the main definitions and theorems. Next, we recall the most widely-expanded control methods for analyzing quantum control problems in four main categories, including open-loop control, closed-loop learning control techniques, feedback control, and robust control. Each method is presented by introducing its specific aspects and recent applications to the field. The paper ends with brief conclusions and an overview on prospective research challenges.

2. Quantum Control Models

In the mathematical formulation of quantum mechanics, the state of a closed quantum system is described by a unit vector $|\psi\rangle$ in a complex Hilbert space $\mathcal{H}$ using the Dirac representation, [63]. A quantum state can be represented by the wavefunction vector $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$, where $\theta \in [0,\pi]$, $\varphi \in [0,2\pi]$, and $|0\rangle$ and $|1\rangle$ correspond to the states zero and one for a classical bit, respectively, [64]. However, the state of open quantum systems or quantum ensembles cannot be expressed as unit vectors and is described through density operator $\rho : \mathcal{H} \to \mathcal{H}$, which is Hermitian, positive, and has trace equal to one. For instance, an ensemble of pure states $\left|\psi_j\right\rangle$ with respective probabilities p_j can be described as $\rho \equiv \sum_j p_j \left|\psi_j\right\rangle\left\langle\psi_j\right|$, where $\left\langle\psi_j\right| = \left|\psi_j\right\rangle^\dagger$ denotes the adjoint, and $\sum_j p_j = 1$. There are several approaches for modelling a quantum system to be controlled. In Schrödinger model of quantum mechanics, bilinear models, including Schrödinger and quantum Liouville equations, are used to express closed quantum systems, while Markovian master equation and stochastic mater equation are used for open quantum systems that are exposed to unavoidable interactions with external environment such as control inputs and measurement devices. However, the adoption of Heisenberg model may be more convenient for some cases. This class of systems are modelled by linear quantum stochastic differential equations. In the sequel, we provide a brief review of some relevant quantum control models.

2.1. Bilinear models

2.1.1. Schrödinger

In quantum mechanics, the evolution of state vector $|\psi(t)\rangle$ can be described by Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H(t)|\psi(t)\rangle, \quad |\psi(t=0)\rangle = |\psi_0\rangle \tag{1}$$

where i is the imaginary unity, $H(t)$ is the quantum-mechanical Hamiltonian of the system, which is a Hermitian operator on $\mathcal{H}$, and $\hbar$ is the reduced Planck's constant, usually considered as a unit for convenience. One has to notice that the term "Hamiltonian" used here is the quantum mechanical Hamiltonian and is not the same as Pontryagin Hamiltonian in dynamic optimization.

The system control can be recognized by a control function set $u_k(t) \in \mathbb{R}$, which is connected to the quantum system through time independent interaction Hamiltonians $H_k (k = 1, 2, \dots)$. Therefore, the controlled evolution is determined by the total quantum mechanical Hamiltonian $H(t) = H_0 + \sum_k u_k(t) H_k$. Here, H_0 indicates the free

Hamiltonian, and H_k is the interaction Hamiltonian. The solution of the Schrödinger equation can be expressed in terms of the unitary time evolution operator $U(t, t_0)$ that can transform the pure state $|\psi(t_0)\rangle$ at some initial time t_0 to the pure state $|\psi(t)\rangle$ at time t, so $|\psi(t)\rangle = U(t, t_0)|\psi(t_0)\rangle$ and the operator equation for the time evolution operator $U(t, t_0)$ is obtained as

$$i\dot{U}(t) = H(t)U(t, t_0), \quad U(t_0) = I \tag{2}$$

where $U(t, t_0)$ is a unitary operator. For an isolated physical system, in which the Hamiltonian is independent of time, equation (2) can easily be combined to yield the famous formulation $U(t, t_0) = e^{-iH(t-t_0)}$. However, for closed quantum systems, in which the system under consideration may be driven by external forces, the dynamics of the system can be formulated in terms of a possibly time-dependent Hamiltonian generator, so the solution of (2) subjected to its initial condition can be expressed as a time-ordered exponential term $U(t) = T_{\leftarrow} \exp\left[-i \int_{t_0}^{t} H(s)ds\right]$, where $T_{\leftarrow}$ denotes the chronological time-ordering operator. For an N-dimensional quantum system, the eigenstates set $\tilde{D} = \{|\phi_i\rangle\}_{i=1}^{N}$ form an orthogonal basis forthe N-dimensional complex Hilbert space $\mathcal{H}$. Therefore, the expansion of the evolving state in terms of eigenstates according to quantum superposition principle, gives $|\psi(t)\rangle = \sum_{j=1}^{N} c_j(t)\left|\phi_j\right\rangle$. Considering $C(t) = (c_1(t), c_2(t), \ldots, c_N(t))$ results in

$$i\dot{C}(t) = H(t)C(t), \quad C(t_0) = \left(\left\langle \phi_j | \psi(t_0) \right\rangle\right)_{j=1}^{N}. \tag{3}$$

For (3), an admissible control set is found to obtain desired coefficient trajectories and achieve the desired state trajectory. Equations (1), (2), and (3) can easily be converted to one other whenever the controllability is satisfied, [63].

2.1.2. Liouville-von Neumann

For systems in mixed states, quantum statistical ensemble can be represented by the statistical density operator $\rho(t) = U(t, t_0)\rho(t_0)U^{\dagger}(t, t_0)$, in which $\rho(t_0) = \sum_{\alpha} w_\alpha |\psi_\alpha(t_0)\rangle\langle\psi_\alpha(t_0)|$ with w_α indicating positive weights and $|\psi_\alpha(t_0)\rangle$ being the normalized state vector evolving in time according to (1). By differentiating $\rho(t)$ with respect to time, Liouville-von Neumann equation,

$$i\dot{\rho}(t) = [H(t), \rho(t)] \tag{4}$$

is obtained, where $\rho(t)$ is the variable to be controlled. Equation (4) may also be written similar to its classical counterpart as $\dot{\rho}(t) = \mathcal{L}(t)\rho(t)$, in which $\mathcal{L}$ is Liouville super-operator. Liouville equation conducts the representation $\rho(t) = T_{\leftarrow} \exp\left[\int_{t_0}^{t} \mathcal{L}(s)ds\right]\rho(t_0)$ for time dependents and $\rho(t) = \exp[\mathcal{L}(t - t_0)]\rho(t_0)$ for time independent Hamiltonians, [65].

2.2. Master Equations

2.2.1. Markovian Model

The evolution of closed quantum systems is unitary, and, therefore, the spectrum of the quantum state is maintained. However, the evolution of open quantum systems cannot be expressed through unitary transformations. Principally, an open quantum system $\mathcal{S}$ is considered to be coupled to an environment $\mathcal{B}$, so the Hilbert space of the total system is represented by $\mathcal{H} = \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{B}}$, where $\otimes$ is the tensor product. The total Hamiltonian may take the form of $H(t) = H_{\mathcal{S}} \otimes I_{\mathcal{B}} + I_{\mathcal{S}} \otimes H_{\mathcal{B}} + \hat{H}_I(t)$, where $\hat{H}_I$ describes the system environment interaction, $I_{\mathcal{B}}$ and $I_{\mathcal{S}}$ denote the identity in the Hilbert space $\mathcal{H}_{\mathcal{B}}$ and $\mathcal{H}_{\mathcal{S}}$, respectively. Hence, the observables referring to $\mathcal{S}$ have the form of $A \otimes I_{\mathcal{B}}$, where A indicates an operator acting on the Hilbert space $\mathcal{H}_{\mathcal{S}}$. The expectation value of the observable A is determined by $\langle A\rangle = tr_{\mathcal{S}}\{A\rho_{\mathcal{S}}\}$, where $\rho_{\mathcal{S}}(t) = tr_{\mathcal{B}}\rho(t)$ is the reduced density operator of the open quantum system $\mathcal{S}$. The notation $tr_{\mathcal{X}}$ denotes the partial trace over the space $\mathcal{X}$. In a similar way, the dynamics of the reduced density matrix $\rho_{\mathcal{S}}$ is described by $\frac{d}{dt}\rho_{\mathcal{S}}(t) = -itr_{\mathcal{B}}[H(t), \rho(t)]$. By neglecting memory effects, the reduced system dynamics can be formulated through a quantum dynamical semigroup. Let consider the dynamical map $V(t) : \mathcal{S}(\mathcal{H}_{\mathcal{S}}) \rightarrow \mathcal{S}(\mathcal{H}_{\mathcal{S}})$,

which is a convex linear completely positive trace preserving (CPTP) quantum operation, expressing the state evolution of the open quantum system. The dynamical map can yield to the representation $V(t)\rho_S = \sum_{\alpha,\beta} W_{\alpha\beta}(t)\rho_S W_{\alpha\beta}^{\dagger}(t)$, where $W_{\alpha\beta}$ are operators in $\mathcal{H}_S$ satisfying $\sum_{\alpha,\beta} W_{\alpha\beta}^{\dagger}(t) W_{\alpha\beta}(t) = I_S$. Also consider the linear map $\mathcal{L}$, which is the generator of the semigroup, regarded as a generalisation of the Liouville super operator. Hence, the semigroup in the exponential form can be represented as $V(t) = \exp(\mathcal{L}t)$ yielding to the Markovian quantum master equation $\frac{d}{dt}\rho_S(t) = \mathcal{L}\rho_S(t)$, known as Lindblad equation, for the reduced density operator of an open system. A general form of $\mathcal{L}$ of a quantum dynamical semigroup can be constructed on a Hilbert space of dimension N with the corresponding Liouville complex space of dimension N^2. A complete basis of orthonormal operators $\{F_i\}_{i=1}^{N^2}$ can be chosen such that $\left(F_i, F_j\right) \equiv tr_S\left\{F_i^{\dagger} F_j\right\} = \delta_{ij}$ with one of the basis operators proportional to the identity and other basis operators traceless, for convenience. Therefore, for an N-dimensional open quantum system with Markovian dynamics, the standard form of the generator is written through the Markovian master equation, [65, 66],

$$\mathcal{L}\rho_S = -i[H,\rho_S] + \sum_{j,k=1}^{N^2-1} a_{jk}\left(F_j\rho_S F_k^{\dagger} - \frac{1}{2}\left\{F_k^{\dagger}F_j,\rho_S\right\}\right) \tag{5}$$

where the Hermitian operator $H = \frac{1}{2i}\left(F^{\dagger} - F\right)$ in which $F = \frac{1}{\sqrt{N}}\sum_{j=1}^{N^2-1} a_{jN^2}F_j$, and the coefficients are defined as $a_{jk} = \lim_{\varepsilon\to 0}\frac{c_{jk}(\varepsilon)}{\varepsilon}$, $j,k = 1,\ldots,N^2-1$ and $c_{jk}(t) \equiv \sum_{\alpha\beta}\left(F_j, W_{\alpha\beta}(t)\right)\left(F_k, W_{\alpha\beta}(t)\right)^*$. Since the matrix coefficient $a = \left(a_{jk}\right)$ is positive, it is possible to diagonalize it by means of a unitary transformation U, such that $UaU^{\dagger} = diag(\gamma_1,\gamma_2,\cdots,\gamma_{N^2-1})$. Here, the eigenvalues γ_j are non-negative and physically specify the relevant relaxation rates, [66]. Therefore, by introducing the new set of Lindblad operators A_l through $F_j = \sum_{l=1}^{N^2-1} u_{lj}A_l$, the diagonal form of a quantum dynamical semigroup can be written down as

$$\mathcal{L}\rho_S = -i[H,\rho_S] + \frac{1}{2}\sum_{l=1}^{N^2-1}\gamma_l\left(\left[A_l\rho_S, A_l^{\dagger}\right] + \left[A_l, \rho_S A_l^{\dagger}\right]\right). \tag{6}$$

It is more convenient to present the *dissipator* as the second term of (6) and rewrite the quantum master equation as

$$\frac{d}{dt}\rho_S(t) = -i[H,\rho_S(t)] + \mathcal{D}(\rho_S(t)). \tag{7}$$

Open systems subjected to external time dependent fields are described via a time dependent generator. A time dependent generalization of Markovian quantum master equation can be expressed as $\frac{d}{dt}\rho_S(t) = \mathcal{L}(t)\rho_S(t)$.

2.2.2. Adjoint Model

The theory of quantum dynamical semigroup is commonly represented by the Heisenberg rather than Schrödinger model. For every system operator A in Hilbert space $\mathcal{H}_S$ in Schrödinger model, it is possible to define a corresponding operator $A_H(t)$ in Heisenberg model related by $tr_S\{A(V(t,0)\rho_S)\} = tr_S\left\{\left(V^{\dagger}(t,0)A\right)\rho_S\right\} = tr_S\{A_H(t)\rho_S\}$ being the Lindblad generator allowed to be time dependent. The adjoint propagator $V^{\dagger}(t,t_0)$ denotes the dynamical map in Heisenberg model defined as $V^{\dagger}(t,t_0) = T_{\rightarrow}\exp\left(\int_{t_0}^{t}\mathcal{L}^{\dagger}(s)ds\right)$, and $\mathcal{L}^{\dagger}$ indicates the adjoint generator defined by $tr_S\{A\mathcal{L}(t)\rho_S\} = tr_S\left\{\left(L^{\dagger}(t)A\right)\rho_S\right\}$. The adjoint propagator satisfies $\frac{\partial}{\partial t}V^{\dagger}(t,t_0) = V^{\dagger}(t,t_0)\mathcal{L}^{\dagger}(t)$ and expresses how operators evolve in time in the Heisenberg model as $A_H(t) = V^{\dagger}(t,0)A$. It follows the adjoint master equation,

$$\frac{d}{dt}A_H(t) = V^{\dagger}(t,0)\left\{\mathcal{L}^{\dagger}\{t\}A\right\}. \tag{8}$$

An important special case for (8) is when Lindblad generator is independent of time as in (6). In this case, the adjoint equation (8) can be simplified to $\frac{d}{dt}A_H(t) = \mathcal{L}^{\dagger}A_H(t)$. Hence, the right-hand side of equation (8) would only depend on $A_H(t)$ at time t, [65].

2.3. Stochastic Master Equation

In feedback control, a quantum system is continuously monitored to obtain feedback information that is used while the measurement proceeds to change the system Hamiltonian in order to obtain the desired behaviour. The state evolution of a system in an infinitesimal time increment dt is described by the stochastic Schrödinger equation

$$d|\psi\rangle = \left\{-\kappa(X - \langle X\rangle)^2 dt + (2\kappa)^{1/2}(X - \langle X\rangle)\, dW\right\}|\psi(t)\rangle \tag{9}$$

where X is an observable with $\langle X\rangle = tr(X\rho)$, κ is a measurement strength parameter, and dW indicates a Wiener increment. The set of measurement increments $dy(t) = \langle X\rangle dt + \dfrac{dW}{(8\kappa)^{1/2}}$ is called the measurement record. It is possible to write (9) in terms of density operator and obtain

$$d\rho = -\kappa[X,[X,\rho]]\, dt + (2k)^{1/2}(X\rho + \rho X - 2\langle X\rangle\rho)\, dW \tag{10}$$

known as the stochastic master equation. One should have in mind that all terms must be kept proportional to dW^2, and we define $\rho(t + dt) \equiv \rho(t) + d\rho$. Since the density operator ρ in (10) uses information from continuous measurements, it is a conditional state. Depending on different measurement processes, stochastic master equation may be written in various forms, [67, 68], however, (10) indicates a typical formula. In general, the existence of the environment memory effect makes the most important difference between Markovian systems and non-Markovian systems.

2.4. Linear Quantum Stochastic Differential Equation

As mentioned before in Heisenberg model, time dependent operators on Hilbert space may be used to describe the quantum dynamics. Let consider a class of non commutative linear stochastic systems described as

$$\begin{aligned} dx(t) &= Ax(t)\, dt + Bdw(t), \qquad x(0) = x_0 \\ dy(t) &= Cx(t)\, dt + Ddw(t) \end{aligned} \tag{11}$$

where A, B, C, and D are real $\mathbb{R}^{n\times n}$, $\mathbb{R}^{n\times n_w}$, $\mathbb{R}^{n_y\times n}$, and $R^{n_y\times n_w}$ matrices, respectively, with positive integer n, n_w, and n_y. $x(t) = [x_1(t)\ldots x_n(t)]^T$ indicates a vector of state variables and x_0 shows the system state variable at initial time. The vector w expresses the input signals admitting the decomposition $dw(t) = \beta_w(t)\, dt + d\tilde{w}(t)$, where $\beta_w(t)$ is a self adjoint, adapted process, and $\tilde{w}(t)$ indicates a vector of self adjoint quantum noises, satisfying $d\tilde{w}(t)\, d\tilde{w}^T(t) = F_{\tilde{w}}dt$, [69], in which $F_{\tilde{w}}$ is a non negative Hermitian matrix. To make sure that the system described by (11) is physically realizable, additional constraints need to be added.

3. Controllability in Quantum Systems

One of the main theoretical and practical notions in the control of quantum systems concerns the possibility of controlling an especial system by means of a proper control signal. The importance of controllability is due to its connection with the universality concept in quantum computing. Various kinds of controllability for quantum systems have been introduced, being some of them briefly presented in this section. Here, the N-dimensional complex unit sphere and unitary group are denoted by S_C^N and $U(N)$ (or $SU(N)$), respectively, with $u(N)$ (or $su(N)$) denoting the corresponding Lie algebra. The Lie algebra generated by the operators $\{-iH_0, -iH_1, \ldots, -iH_K\}$ is mentioned as $\mathcal{L}_0$. To address the existence of controls that can produce the desired results, definitions of controllability and reachability need to be adopted. For the controlled quantum system (1), the state $|\psi(t)\rangle$ in the unit sphere evolves from $|\psi_0\rangle$ on a set M, which forms a differential manifold, finite or infinite dimensional. Therefore, given $|\psi_0\rangle$, $\left|\psi_f\right\rangle \in M$, the state $\left|\psi_f\right\rangle$ is reachable from $|\psi_0\rangle$ at time $t_f > 0$ if there exist an admissible control $u(t)$ such that $\psi\left(t = t_f\,|u,\ \psi_0\right) = \psi_f$. The set of system states that are reachable from $|\psi_0\rangle$ at time t_f, i.e., the set of points in S_C^N reachable at t_f, is denoted by $R_{t_f}(|\psi_0\rangle)$, and the set of system states that are reachable from $|\psi_0\rangle$ at any positive time is denoted by $R(|\psi_0\rangle) \equiv \bigcup_{s>0} R_s(|\psi_0\rangle)$, [70].

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Definition 3.1. *(Complete Controllability) The system is said to be completely controllable if* $R(|\psi_0\rangle) = M$ *is valid for all* $|\psi_0\rangle \in M$*, [10, 71].*

Definition 3.2. *(Operator Controllability) If every desired admissible operation on an arbitrary state can be performed by means of a proper control field, then, the quantum system* (2) *is said to be operator-controllable, [10, 72].*

Definition 3.3. *(State Controllability) The quantum system* (1) *is (pure) state controllable if for every pair of initial* $|\psi_0\rangle$ *and final* $\left|\psi_f\right\rangle$ *states in* S_C^{N-1} *there exist control functions* $\{u_k(t)\}$ *and a time* $t > 0$ *such that the corresponding solution of* (1) *at time t, with initial condition* $|\psi_0\rangle$*, is* $|\psi(t)\rangle = \left|\psi_f\right\rangle$*, [10, 72].*

Definition 3.4. *(Equivalent-state Controllability) The quantum system* (1) *is equivalent state controllable if for every pair of initial* $|\psi_0\rangle$ *and final* $\left|\psi_f\right\rangle$ *states in* S_C^N *there exist controls* $\{u_k(t)\}$ *and a phase factor* $\theta \in [0, 2\pi)$ *such that the solution* $|\psi(t)\rangle$ *of* (1) *satisfies* $|\psi(0)\rangle = |\psi_0\rangle$ *and* $|\psi(t)\rangle = e^{i\theta}\left|\psi_f\right\rangle$*, [10].*

Definition 3.5. *(Eigenstate Controllability) By considering* $|\phi_1\rangle, |\phi_2\rangle, \ldots, |\phi_n\rangle$ *as the n eigenstates of the internal Hamiltonian of an infinite dimensional system, then system* (1) *is strongly eigenstate controllable if* $\bigcup_{i=1}^{n} R_t(|\phi_i\rangle) = S_C^N$ *for all positive t, and* $R_t(|\phi_i\rangle)(i = 1, 2, \ldots, n)$ *is called the eigenstate-from reachable set of eigenstate* $|\phi_i\rangle$*. If* $\bigcup_{i=1}^{n} R(|\phi_i\rangle) = S_C^N$ *holds, then the quantum system is called eigenstate controllable. If a quantum system is pure state controllable, it must also be eigenstate controllable, [73].*

Definition 3.6. *(Density Matrix Controllability) Let consider a pair of unitarily equivalent density matrices* ρ_a *and* ρ_b*. If there exist a control function* $u_k(t)$ *at time* $t > 0$*, such that the solution of* (2) *satisfies* $U(t)\rho_1 U^*(t) = \rho_2$*, the quantum system is density matrix controllable. This notion of controllability is especially useful when a mixed ensemble of states is considered, [10, 71].*

Definition 3.7. *(observable Controllability) If for any observable A and initial density operator* ρ_0*, there exists an admissible control-trajectory pair* $u(t)$ *and* $U(t)$ *defined for* $t_0 \le t \le t_f$ *(with* $t_f < \infty$*), such that the ensemble average of the observable* $tr[\rho(t) A]$ *assumes any kinematically admissible value, the quantum system subjected to Hamiltonian dynamics is observable controllable, [71].*

Definition 3.8. *(Analytical Controllability) Let* $|\psi_0\rangle$ *be an analytic vector from an analytic domain* $\mathcal{D}_w$ *dense in state space. The quantum system in* (1) *is strongly analytically controllable (respectively, analytically controllable) on* $M \subseteq S_C^N$ *if* $R_t(|\psi_0\rangle) = M \cap \mathcal{D}_w$ *holds for all* $t > 0$ *and all* $|\psi_0\rangle \in M \cap \mathcal{D}_w$ *(respectively, if* $R(|\psi_0\rangle) = M \cap \mathcal{D}_w$ *holds for all* $|\psi_0\rangle \in M \cap \mathcal{D}_w$*), [70].*

There already exist several algorithms to test the controllability of some particular quantum systems by using the root space decomposition of $SU(N)$, [74]. The common controllability methods focus on finite dimensional quantum systems for which the controllability can be described by the structure and rank of corresponding Lie groups and Lie algebras leading to a mathematical treatment of closed quantum systems. However, verifying these criteria may be difficult to compute for large dimensional quantum systems. Furthermore, the results obtained this way give only sufficient conditions for exact controllability, so another technique has been proposed based on graph theory for which the criteria on the internal and interaction Hamiltonians are presented to ensure full wavefunction controllability, [75]. By considering (3) with only one single control input $u(t)$ and matrices A and B as internal and interaction Hamiltonians, respectively, we can express that

$$\begin{aligned} & i\hbar\dot{C}(t) = AC(t) + u(t)\,BC(t); \\ & C(t=0) = C_0 \\ & C_0 = \left(c_{0j}\right)_{j=1}^{N}; c_{0j} = \left\langle \phi_j \middle| \psi_0 \right\rangle; \\ & \sum_{j=1}^{N} \left|c_{0j}\right|^2 = 1. \end{aligned} \tag{12}$$

Then, the system is associated to a non-oriented graph $G = (V, E)$, known as the connectivity graph, [76], in which the set of vertices V and the set of edges E correspond to the set of eigenstates $|\phi_i\rangle$, and the set of all pairs of eigenstates

directly coupled by the matrix B, respectively.

$$\begin{aligned} &G(V,E): \\ &V = \{|\phi_1\rangle, \ldots, |\phi_N\rangle\}, \\ &E = \left\{\left(|\phi_i\rangle, \left|\phi_j\right\rangle\right); i < j, B_{ij} \neq 0\right\} \end{aligned} \tag{13}$$

The decomposition of the above-mentioned graph into connected components can be interpreted by $G_\alpha(V_\alpha, E_\alpha)$, $\alpha = 1, \ldots, K$, and $\omega_{kl} = \lambda_k - \lambda_l$ with $k, l = 1, \ldots, N$ denotes the eigenvalue difference for matrix A.

Theorem 3.9. *The system* (12) *is controllable in terms of the connectivity graph, and wavefunction controllability can be assessed if the following three conditions hold, [75]:*

- *(i) The graph G is connected (K=1).*
- *(ii) The graph G does not have degenerate transitions, which means for all* $(i, j) \neq (a, b)$, $i \neq j$, $a \neq b$, *such that* $B_{ij} \neq 0$ *and* $B_{ab} \neq 0$ *then* $\omega_{ij} \neq \omega_{ab}$.
- *(iii) For each* $i, j, a, b = 1, \ldots, N$ *such that* $\omega_{ij} \neq 0$ *then* $\left(\frac{\omega_{ab}}{\omega_{ij}}\right) \in Q$, *in which Q represents the set of all rational numbers.*

However, there are alternative controllability results that completely exclude the last assumption, [77].

Methods to design control laws enabling transitions between given states are very important. However, it is usually not possible to design such control laws from the results obtained from the controllability. Therefore, useful methods need to be developed. One simple scheme to construct an appropriate control law is based on Lie group decomposition, [19]. This method is constructive but may be hard to fully address the decomposition for feasible systems, [78].

4. Open-loop Control

In this method, the control law is completely determined prior to the control experiment, and then used during the experiment without modifications. Coherent open-loop control applies time dependent potentials $\sum_k u_k(t) H_k$ to the system. A quantum system is open-loop controllable if the potentials can be modulated by varying control fields so that the system is driven from some known initial state to the target state.

4.1. Optimal Control

A general optimal control problem can be formulated as follows: Given a set $\mathcal{X}$ of state functions $x : \mathbb{R} \to \mathbb{R}^n$, and a set $\mathcal{U}$ of control functions $u : \mathbb{R} \to \mathbb{R}^m$, find the functions $x \in \mathcal{X}$ and $u \in \mathcal{U}$, which minimize a cost functional $J : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$ and satisfy the dynamical constraint $\dot{x} = f(x, u)$. Virtually, any control problem can be formulated as a particular case of the above-mentioned general optimal control problem, [79]. Therefore, for a system with state vector $x(t)$ controlled by $u(t)$ over the time interval $[0, T]$, the objective functional J in the form of a problem of Bolza, is expressed as

$$J := \Phi(x(T), T) + \int_0^T L(x(t), u(t), t)\, dt \tag{14}$$

with Φ and L smooth functions $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}, \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$, respectively. The task is to minimize J subjected to the condition that the system dynamics is satisfied, with $x(0) = x_0$ and $u(t)$ restricted to the set of admissible controls. By introducing a Lagrange multiplier, the most well-known necessary conditions to minimize (14) are known as Pontryagin's maximum principle (PMP), [14, 15, 79]. Quantum optimal control theory is an extremely powerful tool for achieving quantum control objectives in most practical control problems, [80]. The key concept of optimal control is that the control law will be obtained via minimizing a cost function that drives system (1) from $|\psi(0)\rangle = |\psi_0\rangle$ to

the target $|\psi(T)\rangle = |\psi_d\rangle$. From now onwards, the problem of determining the control fields $u \in L^2(\mathbb{C}, [0,T])$ while satisfying equation (1) is considered. In this context, a general cost functional can be expressed as, [81],

$$J(|\psi\rangle, u) = \frac{1}{2}\left\||\psi(T)\rangle - |\psi_d\rangle\right\|^2_{\mathbb{C}^n} + \frac{\gamma}{2}\|u\|^2_{L^2(\mathbb{C},[0,T])} + \frac{1}{2}\sum_{j=1}^{n}\alpha_j\left\|\psi_j\right\|^2_{L^2(\mathbb{C},[0,T])} \tag{15}$$

with the weighting factors $\gamma > 0$, $\alpha_j \geq 0$. It is obvious that further constraints can also be added in the same way. The problem of optimal control for minimizing cost functional (15) subject to equation(1) under normal conditions admits a solution $(\bar{\psi}, \bar{u}) \in H^1(\mathbb{C}^n, [0,T]) \times L^2(\mathbb{C}, [0,T])$, [82]. In order to compute the first order necessary optimality conditions, the method of Lagrange multipliers is used, so the Lagrangian function is defined as

$$L(|\psi\rangle, |p\rangle, u) = J(|\psi\rangle, u) + \mathrm{Re}\left\langle p, i\dot{\psi} - H\psi\right\rangle \tag{16}$$

where $\langle\phi, \psi\rangle = \int_0^T \phi.\psi^* dt$, in which the symbols "*" and "." mean the complex conjugate and the typical vector-scalar product in $\mathbb{C}^n$, respectively. The necessary conditions for the minimization problem

$$L\left(\left|\tilde{\psi}\right\rangle, |\tilde{p}\rangle, \tilde{u}\right) = \inf_{|\psi\rangle \in X_t^0, |p\rangle, u \in X_t} L(|\psi\rangle, |p\rangle, u) \tag{17}$$

in which $X_t = L^2(\mathbb{C}^n, [0,T])$ and $X_t^0 = X_t \bigcap \{\psi : \psi(0) = \psi_0\}$, are derived by equating to zero the Fréchet derivative of L with respect to the triple $(|\psi\rangle, |p\rangle, u)$. Therefore, the optimality system

$$\begin{aligned}
&\left|\dot{\psi}\right\rangle = -i(H_0 + H_k u)|\psi\rangle, |\psi(0)\rangle = |\psi_0\rangle \\
&|\dot{p}\rangle = -i\left(H_0^* + H_k u\right)|p\rangle - q, \\
&|p(T)\rangle = -i(|\psi(T)\rangle - |\psi_d\rangle) \\
&u = \frac{1}{\gamma}\mathrm{Re}\left[|p\rangle\left(\frac{\partial(H_0 + H_k u)}{\partial u_{re}}|\psi\rangle\right)^*\right] + i\frac{1}{\gamma}\mathrm{Re}\left[|p\rangle\left(\frac{\partial(H_0 + H_k u)}{\partial u_{im}}|\psi\rangle\right)^*\right]
\end{aligned} \tag{18}$$

is obtained, where $q_j = \alpha_j\psi_j$ and $u = u_{re} + iu_{im}$, [19]. Once optimal control formulas are established, it is possible to formulate a numerical algorithm to solve the optimality equations for given initial and final states, [83, 84]. For instance, iterative algorithms such as Krotov-type, [85], and GRAPE-type, [86], have been used to generate controls. A detailed comparison between these two schemes can be seen in [87, 88]. Moreover, based on the explicit calculation of the extremal solutions of the PMP, there exist analytical results applicable for two and three level quantum systems, [89], (to see a comparison between two- and three-level models, refer to [90]). The following indicates some applications of GRAPE algorithm. Given H_0 and H_k specifying the quantum system, the objective of GRAPE algorithm is then to seek optimal controls $u_k(t)$ in order to perform a desired task. The gradient-based learning method can also be extended to the optimal control of following problems:

4.1.1. Pure State Transformation

The example of steering a pure quantum system from an initial state $|\psi_0\rangle$ to a desired target state $|\psi_d\rangle$ over a time interval $[0,T]$ is considered. The performance function to be maximized is the quantum fidelity determining the transition probability between $|\psi_d\rangle$ and $|\psi(T)\rangle$ by the modulus of their overlap as $\phi_1 := \mathrm{Re}\{\langle\psi_d|\psi(T)\rangle\}$. The controls are restricted to a piecewise-constant form, so the evolution can be decomposed into M time slices of length $\Delta t = T/M$ with the Hamiltonian of the j'th slice as $H^{(j)} = H_0 + \sum_{k=1}^{m} u_k^{(j)} H_k$, which is constant over every Δt. Therefore, the unitary propagator $U(T)$ can be obtained via direct integration of (2) yielding $U(T) = U_M U_{M-1} \ldots U_2 U_1$ with $U_j = \exp\left\{-i\Delta t H^{(j)}\right\}$, so the quality function can be rewritten as $\phi_1 = \mathrm{Re}\{\langle\psi_d|U_M U_{M-1} \ldots U_2 U_1|\psi_0\rangle\}$. The derivative

of this with respect to $u_k^{(j)}$ is $\frac{\partial \phi_1}{\partial u_k^{(j)}} = \mathrm{Re}\left\{\left\langle \psi_d \left| U_M U_{M-1} \ldots U_{j+1} \frac{\partial U_j}{\partial u_k^{(j)}} U_{j-1} \ldots U_2 U_1 \right| \psi_0 \right\rangle\right\}$, in which

$$\frac{\partial U_j}{\partial u_k^{(j)}} = -i\left(\int_0^{\Delta t} U_j(\tau) H_k U_j(-\tau) d\tau\right) U_j \approx -i\Delta t H_k U_j \tag{19}$$

where $U_j(\tau) = \exp\left\{-i\tau H^{(j)}\right\}$, and for all time slices the bound $\Delta t \ll \left\|H^{(j)}\right\|^{-1}$ is considered. By substitution, the approximate gradient is given as

$$\frac{\partial \phi_1}{\partial u_k^{(j)}} \approx \mathrm{Re}\left\{-i\Delta t \left\langle \psi_c \left| U_M \ldots U_{j+1} H_k U_j \ldots U_1 \right| \psi_0 \right\rangle\right\} \tag{20}$$

The performance function will increase by choosing

$$u_k^{(j)} \to u_k^{(j)} + \varepsilon \frac{\partial \phi_1}{\partial u_k^{(j)}}, \tag{21}$$

where ε is a small positive value. Gradient formula (20) form the foundation of the GRAPE algorithm. For this example of direct ascent of ϕ_1, the algorithm can be outlined as follows:

(i) Set the $M \times m$ initial control amplitudes $u_k^{(j)}$.

(ii) Calculate the forward-propagated states as
$\left|\psi_j\right\rangle := U_j \ldots U_1 |\psi_0\rangle$ for $j = 1, \ldots, M$.

(iii) Calculate the backward-propagated states as
$\left|\lambda_j\right\rangle := U_{j+1}^{\dagger} \ldots U_M^{\dagger} |\psi_c\rangle$ for $j = 1, \ldots, M$.

(iv) Calculate (20) and update according to (21).

(v) With the new controls, go to the second step.

With sufficiently small step size at each iteration and exact gradients, (see [91]), the algorithm is guaranteed to converge monotonically to a local maximum for performance function, [92]. As seen before, for cases in which controls are smooth continuous functions, a gradient formula of the same form as (20) can be obtained quickly from PMP.

4.1.2. Unitary Gate Synthesis

For some applications, it is necessary to implement unitary gate synthesis. The performance function for obtaining the maximum overlap with a target gate U_c is defined as $\phi_2 := \left|tr\left\{U_c^{\dagger} U(T)\right\}\right|^2$, and gradients are obtained as

$$\frac{\partial \phi_2}{\partial u_k^{(j)}} \approx 2\,\mathrm{Re}\left\{-i\Delta t \left\langle P_j \left| H_k X_j \right\rangle \left\langle X_j \left| P_j \right\rangle\right\}\right. \tag{22}$$

with $X_j := U_j \ldots U_1$ and $P_j := U_{j+1}^{\dagger} \ldots U_M^{\dagger} U_c$.

4.1.3. Subspace Transformation

In subspace to subspace transfer, any number d of initial and target states may be specified through the performance function

$$\phi_3 := \left|\left\langle \psi_c^{(1)} | U(T) | \psi_0^{(1)} \right\rangle + \cdots + \left\langle \psi_c^{(d)} | U(T) | \psi_0^{(d)} \right\rangle\right|^2. \tag{23}$$

The gradients of ϕ_3 are a trivial variation of those for unitary gate synthesis, [92]. In order to reduce relaxation effects, it is very important to apply the control. In other words, it is desirable to recognize the minimum time

required to perform a quantum computation by means of a set of physical resources, [93]. The two-level quantum systems have important applications, amongst them the quantum state transformation is of high importance, [94], which should typically be performed in the shortest possible time. However, the control time is inversely proportional to the control amplitude. Hence, to achieve a very fast state transfer, one needs a very large control amplitude, which is not practically possible since the control signal has to be bounded. There are several researches on time-optimal control of quantum systems including the state transfer problem for two-level systems using the geometric control in the x–y plane, [95], the time optimal transfer of state with bounded bang-bang control, [96], and time-optimal control of population transfer for two-level quantum systems with bounded control, [97].

4.2. Geometric Control

The same problem as [98] for a spin 1/2 system in a constant field along z-axis controlled through radio-frequency (RF) pulses producing an x rotation of the spin is considered. By ignoring the global phase, the initial state $|\psi_0\rangle$ and target state $|\psi_d\rangle$ can be denoted by

$$\begin{aligned} |\psi_0\rangle &= \cos\frac{\theta_0}{2}|0\rangle + e^{i\varphi_0}\sin\frac{\theta_0}{2}|1\rangle \\ |\psi_d\rangle &= \cos\frac{\theta_d}{2}|0\rangle + e^{i\varphi_d}\sin\frac{\theta_d}{2}|1\rangle \end{aligned} \tag{24}$$

where $\theta_0, \theta_d \in [0, \pi]$ and $\varphi_0, \varphi_d \in [0, 2\pi]$. Given any target unitary transformation $U_f \in SU(2)$ satisfying $|\psi_d\rangle = U_f|\psi_0\rangle$, there exist a unique $\beta \in [0, \pi]$ determining the minimum time in order to accomplish U_f such that $U_f = \exp\left(\frac{-i\alpha}{2}\sigma_x\right)\exp\left(\frac{-i\beta}{2}\sigma_z\right)\exp\left(\frac{-i\gamma}{2}\sigma_x\right)$, with $\alpha, \gamma \in \mathbb{R}$. β can be calculated, [99], as

$$\beta = |\arccos(\sin\theta_2\cos\varphi_2) - \arccos(\sin\theta_1\cos\varphi_1)|\cdot \tag{25}$$

The condition holds for a time-optimal transition with unbounded control. For a more general case of a two-level system, the geometric control, [94, 22], can be considered as the following decomposition in x-y plane

$$\begin{cases} \Omega_x(t) = M\cos(2Et+\varphi) \\ \Omega_y(t) = -M\sin(2Et+\varphi) \end{cases} \tag{26}$$

where M indicates the amplitude, and $\varphi \in [0, 2\pi)$ is the initial phase of the controls. Therefore, the total Hamiltonian is

$$H(t) = H_0 + H_k(t) = \begin{pmatrix} -E & Me^{i(2Et+\varphi)} \\ Me^{-i(2Et+\varphi)} & E \end{pmatrix} \tag{27}$$

in which $H_0 = diag(E_1, \ldots, E_n)$, with E_j real numbers expressing the energy levels such that $\sum_{j=1}^{n} E_j = 0$, $H_1 = \sigma_x$, and $H_2 = \sigma_y$. From the Schrödinger equation (1), state evolution operator can be deduced as

$$U(t) = \begin{pmatrix} e^{iEt} & 0 \\ 0 & e^{-i(Et+\varphi)} \end{pmatrix}\begin{pmatrix} \cos(Mt) & i\sin(Mt) \\ i\sin(Mt) & \cos(Mt) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{pmatrix}. \tag{28}$$

For known initial $(\theta_0, \varphi_0, 1)$ and target $(\theta_d, \varphi_d, 1)$ points on the Bloch Sphere, the minimum time transfer under geometric control would be $t_{\min} = \dfrac{\Delta\theta}{2u_{\max}}$, with $u_{\max}$ indicating the maximum amplitude of the control, [19].

4.3. Bang-bang Control

It is sometimes reasonable to assume an unbounded control. However, the control $u(t)$ is generally bounded. Bang-bang control is a simple control strategy in which the control $u(t)$ is modulated between two values $\pm V$ ($V \geq 0$). For instance, consider the time optimal control problem for $i\dot{U}(t) = \frac{1}{2}[\sigma_z + u(t)\sigma_x]U(t)$ to drive the system from $|0\rangle$ to $|\psi_d\rangle$ with bounded control $|u(t)| \leq V$. From the Liouville equation (4) for the evolution of the density matrix $\rho(t)$, with $\rho = \frac{1}{2}(I + r.\sigma)$, in which I indicates the identical matrix, $r = \left(tr\{\rho\sigma_x\}, tr\left\{\rho\sigma_y\right\}, tr\{\rho\sigma_z\}\right)$ the Bloch vector, and

$\sigma = \left(\sigma_x, \sigma_y, \sigma_z\right)$ the Pauli matrices, we have

$$\dot{r}_t = \begin{pmatrix} \dot{x}_t \\ \dot{y}_t \\ \dot{z}_t \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -u(t) \\ 0 & u(t) & 0 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} \tag{29}$$

with $r_0 = (0, 0, 1)$ corresponding to $|0\rangle$. From the performance measure $J(u) = \int_0^t d\tau$ of time optimal control problem and Lagrange multiplier vector, the Hamiltonian function would be $\mathbb{H} \equiv 1 + \lambda^T(t).\dot{r}_t$. According to the PMP, [100], a necessary condition for optimal control to minimize the functional $J(u)$ is $\mathbb{H}\left(r^*(t), u^*(t), \lambda^*(t), t\right) \leq \mathbb{H}\left(r^*(t), u(t), \lambda^*(t), t\right)$, so the optimal control should be chosen as $u^*(t) = -Vsgn\left(\lambda_{.3}(t)\, y_t - \lambda_{2.}(t)\, z_t\right)$, [96, 14, 15]. In addition to the discussed time optimal problem, bang-bang control has also been used for minimal switches, [101].

4.4. Incoherent Control

It is sometimes assumed that the control directly influence the system dynamics. However, there are several physical situations, such as a finite dimensional open quantum system with Markovian dynamics, [102], where it may not be possible or very hard to directly control the state of a system S, but it is easy to manipulate the state of an ancilla system P. Then, the state of S can be modified via interaction with P to control the quantum system. The ancilla system can be, for instance, a probe, which is entangled with the quantum system. The information acquisition about either the interaction, or the input state can be done by measuring the expectation values of the probe, [103].

This kind of incoherent control does not have any effects on the Hamiltonian of S, but it is performed through control on the probe and interaction of the probe with the system, [104]. In an environment-mediated control protocol, under assumption that the dynamics of S and P are only due to the interaction with environment, it has been proved that the induced correlations between S and P are rich enough to allow total control of S by means of P, [105]. Moreover, for large quantum systems, the control process may be done by acting on a small subsystem allowing to transfer arbitrary and unknown quantum states for downloading and uploading accesses useful for the control of quantum RAM and hard drives, [106]. Incoherent resources may also be introduced through quantum measurements. measurement and dissipation provide a unique quantum resource, owing to the stochastic back-action that necessarily accompanies acquisition of information. For instance, it is possible to reach a desirable quantum state with combinations of free quantum evolution and quantum measurement, [107]. In the Strocchi map phase-space, undisturbed time-evolution is a smooth Hamiltonian dynamics, while measurements correspond to jumps in the phase-space. When the Strocchi map is used to describe quantum evolution of the N-level system (1), the phase-space has dimension $2N$ with coordinates (q_k, p_k). The set of required transformations to ensure controllability is the $U(N)$ subgroup of $SO(2N)$ corresponding to the real linear canonical transformations, that also preserve the form $G(\psi', \psi) = \frac{1}{2}\sum_k \left(q'_k q_k + p'_k p_k\right)$. Take $\mathcal{A} = \{H_0, H_1, \ldots, H_r\}_{LA}$ a proper subalgebra of $u(N)$. Then, $S_{\mathbb{C}}^{N-1}$ is a fibre space with the orbits of $G(\mathcal{A})$ as fibres and $U(N)/G(\mathcal{A})$ as base space. Hence, a desired state $|\psi_d\rangle$ can only be reached from the initial state $|\psi_0\rangle$ if they both belong to the same fibre. The system is not purely controllable, but can be incoherently controllable in a sense that given any $|\psi_d\rangle$, there exist a family of observables $M(|\psi_d\rangle)$ such that measurement of one of the observables on any $|\psi_0\rangle$ plus unitary evolution $\int \exp\left(i\tau H(\tau)\right)d\tau$ leads to $|\psi_d\rangle$ if $G(\mathcal{A})$ is either $O(N)$ or $Sp(\frac{1}{2}N)$, [107]. Moreover, measurement-based incoherent control extracts entropy from a system that can be used for detection and correction of errors or imperfections. Measurement-only manipulation has been demonstrated using a fixed measurement basis. Such capability is an additional degree of freedom for measurement-based protocols, including state manipulation, [108, 109, 110], rapid state purification, [111], measurement-based quantum computation, [112] and quantum Zeno dynamics, [113, 114].

5. Closed-loop Learning Control

Closed-loop learning control, [35], (also referred to as adaptive feedback control) is the approach of exploring a quantum control landscape for seeking the optimal control strategy where the objective function reaches its maximum or minimum, [115]. The control objective is typically formulated as an optimal control problem by transforming the

problem into that of optimizing a functional of the quantum states, control inputs, control time, and, possibly, other variables of interest. In the learning procedure, an optimal control problem is resolved by iterations. In the first stage, the external control (for instance a shaped laser pulse) is applied to the system under control (for instance an ensemble of molecules), and the result is observed. Then, the signal is recognized and fed back to the learning process to suggest a better control input according to the privious experiments. Next, the algorithm evaluates each control based on its measured outcome with respect to a predefined control goal, and looks through the space of available controls to proceed with an optimal solution, [36]. The method has been especially successful in controlling the laboratory quantum phenomena, [11, 36], where it is hard to directly design optimal control strategy.

5.1. Gradient-based Learning Methods

In order to achieve scalable quantum computing technologies, gradient-based optimal control algorithms, such as GRAPE, have been widely utilized especially for the purpose of gate implementation. These methods can normally present good performance for solving optimal control problems when the model of controlled system is known, and the dynamics can be equivalently (or approximately) expressed through a closed quantum system, [116]. This has also been analysed by quantum control landscape theory, [117]. Despite the fact that the performance of gradient algorithms is often hindered by deterministic or random errors in the system model and control electronics, these algorithms have been good candidates to numerically find optimal fields. However, by means of the fundamental idea of gradient-based learning control, some variants have been developed for a variety of requirements in quantum optimal control. For instance, the data-driven gradient optimization algorithm (d-GRAPE) is able to correct all deterministic gate errors, with a mild efficiency loss, by jointly learning from a design model and the experimental data from quantum tomography, [118]. A gradient-based frequency domain quantum optimal control method has been proposed to optimize time domain ultrafast laser pulses subject to multiple internal limitations as well as external constraints, [119]. As mentioned before, the objective of quantum control is the manipulation of small scale dynamics phenomena. Hence, it is important to understand the dynamical control within a family of related quantum systems. Concerning this, diffeomorphic modulation under observable-response-preserving homotopy search algorithm, called D-MORPH, has been implemented through a well-developed gradient-based procedure, [120]. Consider the Hamiltonian $H(s,t) = H_0(s) - \mu(s) E(s,t)$, where $H_0(s)$ indicates the free Hamiltonian, $\mu(s)$ is the dipole moment operator, and $E(s,t)$ is the externally applied laser field. The evolution of state $|\psi(s,t)\rangle$ is prescribed by the Schrödinger model (1) with the initial condition $|\psi(s,0)\rangle = |\psi_0\rangle$. The objective is taking $H(s,t)$ along a diffeomorphic path of Hamiltonians, parametrized by s, and identify the associated progression of the electric field $E(s,t)$ such that $\langle O(s)\rangle_T = \langle\psi(s,T)|O|\psi(s,T)\rangle = \langle O(0)\rangle_T$ for some particular target observable operator O. Achieving this goal corresponds to find the solutions $E(s,t)$, to the nonlinear equation $F(s) = \langle O([E(s,t), H_0(s), \mu(s)], T)\rangle - C_T = 0$. However, this equation cannot be solved explicitly because it is highly nonlinear in $E(s,t)$. Local search techniques like the quasi-Newton and conjugate gradient methods, as usually invoked in OCT calculations, require repetitive, separate solutions for different s values and, as a result, yield little insight on how the laser fields are related while traversing a curve in Hamiltonian space. Therefore, a path-following morphing method capable of finding multiple solutions is adapted in [120]. Each solution is associated to a homotopy path, [121], along which the function $F(s)$ is identically zero. Each homotopy path can be computed by integrating the differential equation $\frac{dF(s)}{ds} = \frac{d\langle O(s)\rangle_T}{ds} = 0$ over s starting from $\langle O(s=0)\rangle_T = C_T$ and in conjunction with Schrödinger equation (1).

5.2. Stochastic Searching Methods

Despite the success of gradient algorithm for finding optimal controls numerically, it is sometimes hard to acquire the gradient information or there may be local optima in complex quantum control problems. In these cases, it is a good idea to use stochastic search algorithms. These methods have also been used in the concept of experimental adaptive feedback control (AFC), [36]. The large class of evolutionary algorithms includes simulated annealing, [122], ant-colony systems, [123], memetic algorithms space, [124], Differential evolution (DE), [31, 125, 126], particle swarm optimization (PSO), [127], Genetic algorithm (GA), [35, 128, 30, 129], and etc. However, amongst the many methods proposed, the three that are very similar and popular are the GA, PSO, and DE. Although GA is a commonly used algorithm, PSO and DE algorithms have attracted more attention especially for continuous optimization problems. The promising nature of PSO and DE has been shown in many studies showing the superiority of DE and PSO

over other evolutionary algorithms, [130, 131, 132]. In evolutionary computation methods, crossover, mutation and selection operations are iteratively implemented to search for optimal controls in a parameter space, [116].

In GA, parent chromosomes go through three steps, including selection, crossover, and mutation to generate a new generation (offspring) of chromosomes, [131]. GA methods have been applied to various quantum control problems. For instance, a methodology for teaching laser pulse sequences to excite specified molecular states has been simulated through GA, where the GA code runs on the controlling computer, supplying pulse sequences to the laser and receiving some observable function of the molecular state from the measurement device. Over several generations, the system as a whole will seek to optimize the fields, [35]. In [30], instead of optimizing the electric field by optimal control theory, the transmittance and phase shift of conventional pulse shapers were optimized in the frequency domain by a genetic algorithm to optimize the pulse shape for rovibrational wave-packet manipulation. Control and optimization prospects in the frequency domain are also studied theoretically via GA and shaping Fourier-limited pulses, [123]. Moreover, GA has also been used for the optimal laser control of molecular orientation, [124].

In PSO, the particle evolves according to a Langevin equation that includes a random kick, an attractive force to its previous best fitness, and a force pulling to the particle to the fittest particle in its neighbourhood. Neighbourhoods overlap such that they do not partition into distinct sets, [131]. PSO, [127], has similarities with the genetic algorithm since it also begins with a random population and searches for optima by updating the population. However, PSO has no evolution operators such as the crossover and mutation. Comparing with GA, PSO's advantages lies on its easy implementation and few parameters to adjust. In the quantum particle swarm optimization (QPSO) algorithm, a particle is defined based on the qubit, [133]. This algorithm is a global-convergence-guaranteed and has a better search ability than the original PSO. However, Diversity-Controlled QPSO (DCQPSO) is a more efficient proposed method of controlling the diversity for enabling particles to escape the sub-optima more easily and has been proved to outperform the PSO and QPSO, [134]. In DCQPSO a threshold value is set for population's diversity measure to prevent premature convergence and therefore enhance the overall performance of the QPSO.

In DE, each chromosome breeds with three other randomly chosen chromosomes from the same generation to produce a daughter, and the fittest of the original vs the bred daughter survives to the next generation. In each generation, the difference between two randomly chosen target vectors is weighted, then added to a third randomly selected target vector to generate the new set of vectors called donors. This quantity determines the DE step size for exploring the control landscape. Donor vector elements are incorporated into target vectors to generate trial vectors, and the fittest of the target and trial vectors survive to the next generation, [131]. DE, [31, 125, 126, 135], is a simple but competitive algorithm for real-parameter optimization problems with various variants and applications have appeared in [136, 137]. The mutation strategy is crucial for the efficiency of DE. A direction averaged Differential Evolution (daDE) algorithm has been applied to the tasks of quantum state preparation and quantum gate implementation resulting in several remarkable features including simplicity and robustness, [138]. A subspace-selective self-adaptive differential evolution (SUSSADE) algorithm has been proposed to achieve a high-fidelity single-shot Toffoli gate, [139], and single-shot three-qubit gates, [140]. To enhance the learning performance, an improved DE algorithm with equally-mixed strategies in the training step of the control design has been introduced for open quantum systems. Numerical results confirm the effectiveness of the proposed equally-mixed strategies DE (EMSDE) algorithm regarding the control design for open quantum systems with uncertainties, [125]. In comparative studied amongst several promising evolution algorithms, it has been found that DE usually outperformed GA and PSO for hard quantum control problems, [131].

5.3. Reinforcement Learning Methods

Reinforcement learning (RL), [33], is an active area in machine learning used to understand and automate goal-directed learning and decision-making. Related algorithms and techniques have been used in a variety of applications ranging from artificial intelligence, particularly in robotics, [141], to solve complex problems with incomplete information by using Grey RL (GRL), [142], and the control of nonlinear systems, [143]. Although RL algorithms feature high performance, practical applications may face problems in achieving a good trade off between exploration and exploitation, and also in the speed of learning. Many methods have been proposed to tackle these problems. For instance, temporal abstraction, [144], and decomposition, [145], methods have been explored to address these issues.

Moreover, several learning paradigms has been combined to optimize RL. For instance, regarding the representation and generalisation of continuous action spaces in RL problems, a model has been proposed for representation and generalization in model-less RL based on the self-organising map (SOM) of Kohonen, [146], and Q-learning, [147],

which allows either the one-to-one, many-to-one or one-to-many structure of the desired state–action mapping to be captured [148]. However, the model possessed limitations including issues pertaining to scalability. An adaptation of Watkins' Q-learning, [149], for fuzzy inference systems where both the actions and the Q-functions are inferred from fuzzy rules has been proposed and compared effectively with genetic algorithm, [129], on the cart-centering problem [150]. A fuzzy reinforcement learning (FRL) scheme based on the principles of sliding-mode control and fuzzy logic has been proposed using only immediate reward, [151]. A dynamic fuzzy Q-learning method (DFQL), capable of generating and tuning fuzzy rules, has been presented in [152]. DFQL generalizes the continuous input space with fuzzy rules. In addition, a fuzzy RL algorithm for optimal decision making in highly stochastic problems with long renewal periods and long-lasting memory of actions that have very large state and action spaces has been proposed named ACFRL-2, [153]. Moreover, many other specific improvements are also implemented to improve RL methods, [141, 154, 155]. Rapid development of quantum information processing and computation led to an approach to speed up learning process, [156], in a sense that quantum computation can accelerate the solution of classical problems and even solve the difficult problems unable to be solved by classical algorithms. Shor's factoring algorithm, [157], provides an exponential speedup for factoring large integers into prime numbers, and its experimental demonstration has been realized using nuclear magnetic resonance (NMR), [158]. Besides, Grover's searching algorithm, [159], can achieve a square speedup over classical algorithms in unsorted database searching and has been experimentally implemented through quantum optics, [160], and NMR, [161]. Some methods have also been proposed to connect machine learning and quantum computing. For instance, quantum-inspired evolutionary algorithms have been proposed to improve the existing evolutionary algorithms, [162]. Moreover, quantum computation has been used for parallelization of a fuzzy-logic control algorithm to quicken the fuzzy inference, [163]. A quantum optimization algorithm for combinatorial optimization using the cost structure of the search states has been proposed, and its behaviour has been illustrated for over constrained satisfiability (SAT) and asymmetric travelling salesman problems (ATSP), [164]. In addition, quantum search technique has been used to stochastic planning, [165].

The concept of quantum reinforcement learning (QRL) has been suggested based on quantum computing concepts aiming at improving the trade-off between exploration and exploitation and also speed-up learning. A framework of value-updating algorithm has been introduced inspired by state superposition principle and quantum parallelism. The state $\{s_i\}$ (action $\{a_i\}$) in traditional RL is looked upon as the eigenstate $\{|s_i\rangle\}$ (eigen action $\{|a_i\rangle\}$) in QRL, the (state, action) pair set is represented by the quantum superposition state, and the eigenstate (eigen action) is obtained by randomly observing the simulated quantum state according to the collapse postulate of quantum measurement. The effectiveness and superiority of the QRL algorithm has been shown for complex problems, [34].

In the light of the existing difficulties in action selection, a lot of research has been developed being a method based on quantum computation proposed. The action $\{a_i\}$ represents the possible operation (or control) that can accomplish the state transition between two states. The state $\{s_i\}$ can represent $|\phi_{l_i}\rangle$ and the action $\{a_i\}$ represent the control function that can drive $|\phi_{l_i}\rangle$ to $|\phi_{l_{i+1}}\rangle$. A quantum-inspired reinforcement learning (QiRL) algorithm is proposed for navigation control of autonomous mobile robots, [166]. It has also been shown that the performance of cutting-edge RL techniques is comparable to optimal control methods in the task of finding short, high-fidelity driving protocol from an initial to a desired state in non-integrable many body quantum systems of interacting qubits, [167]. In [168], simultaneously optimization of speed and fidelity of quantum computation against both leakage and stochastic control errors have been done through deep reinforcement learning. A universal control cost function optimization (UFO) framework has been presented. The UFO cost function is used as a reward for a continuous variable policy-gradient RL agent trained by trusted region policy optimization to find highest-reward/minimum cost analogue controls for a variety of two-qubit unitary gates. A network-based agent in RL can discover complete quantum-error-correction strategies, protecting a collection of qubits against noise. This has been shown by utilizing two-stage learning with teacher and student networks and a reward quantifying the capability to recover the quantum information stored in a quantum system. The strategies of quantum-error-correction require feedback adapted to measurement outcome, [169].

5.3.1. Reinforcement Learning and Exploration Strategy

Standard RL framework is based on discrete-time, finite Markov decision processes (MDPs) by assuming that state S and action A_{s_n} can be organized into discrete values, [33]. At a certain step, the agent observes the state of the environment (inside and outside of the agent) S_t and, then, chooses an action a_t. After executing the action, the agent receives a reward $r_t + 1$ reflecting how good that action is (in a short-term sense). The goal of reinforcement learning

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is to learn a mapping from states to actions, i.e., the agent is to learn a policy $\pi : S \times \cup_{i\in S} A_{(i)} \rightarrow [0,1]$, so that the expected sum of discounted reward of each state will be maximized.

$$V^{\pi}_{(s)} = E\left\{r_{t+1} + \gamma r_{t+2} + \gamma^2 r_{t+3} + \cdots | s_t = s, \pi\right\} = E\left[r_{t+1} + \gamma V^{\pi}_{(s_{t+1})} | s_t = s, \pi\right] = \sum_{a\in A_s} \pi(s,a)\left[r^a_s + \gamma \sum_{s'} p^a_{ss'} V^{\pi}_{(s')}\right] \quad (30)$$

in which $\gamma \in [0,1)$ is discounted factor, $\pi(s,a)$ is the probability of selecting action a according to state s under policy π, $p^a_{ss'} = \Pr\{s_{t+1} = s' | s_t = s, a_t = a\}$ is the probability for state transition, and $r^a_s = E\{r_{t+1} | s_t = s, a_t = a\}$ is the expected one-step reward. Then, the optimal state-value function is

$$\begin{aligned} V^*_{(s)} &= \max_{a\in A_s}\left[r^a_s + \gamma \sum_{s'} p^a_{ss'} V^*_{(s')}\right] \\ \pi^* &= \arg\max_{\pi} V^{\pi}_{(s')}, \forall s \in S \cdot \end{aligned} \quad (31)$$

The equation of $V^*_{(s)}$ is also called Bellman equation of V^* in dynamic programming. Note that the V function states what the expected overall value (not reward) of a state s under the policy π is, while the Q function states what the value of a state s and an action a under the policy π is. There are similar value functions and Bellman equations, where $Q^{\pi}_{(s,a)}$ stands for the value of taking action a in state s under policy π, [170],

$$\begin{aligned} Q^{\pi}_{(s,a)} &= E\left\{r_{t+1} + \gamma r_{t+2} + \gamma^2 r_{t+3} + \cdots | s_t = s, a_t = a, \pi\right\} = r^a_s + \gamma \sum_{s'} p^a_{ss'} V^{\pi}(s') = r^a_s + \gamma \sum_{s'} p^a_{ss'} \sum_{a'} \pi(s',a') Q^{\pi}_{(s',a')} \\ Q^*_{(s,a)} &= \max_{\pi} Q_{(s,a)} = r^a_s + \gamma \sum_{s'} p^a_{ss'} \max_{a'} Q^*_{(s',a')} \cdot \end{aligned} \quad (32)$$

5.3.2. Quantum Reinforcement Learning

Similar to traditional RL, a QRL system can also be identified for the three main sub elements, including a policy, a reward function, and a model of the environment (maybe not explicit). However, QRL algorithms are remarkably different from all those traditional RL algorithms in some intrinsic aspects, such as representation, policy, parallelism, and updating operation. Let N_s and N_a be the number of states and actions, then choose m and n in such a way that $N_s \leq 2^m \leq 2N_s$ and $N_a \leq 2^n \leq 2N_a$. These inequalities guarantee that every states and actions in traditional RL have corresponding representation with eigenstates and eigenactions in QRL. Moreover, m and n qubits are used to demonstrate eigenstate set $S = \{|s_i\rangle\}$ and eigenaction set $A = \{|a_i\rangle\}$, respectively. Hence, we can express

$$\begin{aligned} \left|s^{(N_s)}\right\rangle &= \sum_{i=1}^{N_s} C_i |s_i\rangle \\ \left|s^{(m)}\right\rangle &= \sum_{s=\overbrace{00,\ldots,0}^{m}}^{\overbrace{11,\ldots,1}^{m}} C_s |s\rangle, \end{aligned} \quad (33)$$

and

$$\begin{aligned} \left|a^{(N_a)}_{s_i}\right\rangle &= \sum_{i=1}^{N_a} C_j \left|a_j\right\rangle \\ \left|a^{(n)}_{s}\right\rangle &= \sum_{a=\overbrace{00,\ldots,0}^{n}}^{\overbrace{11,\ldots,1}^{n}} C_a |a\rangle \cdot \end{aligned} \quad (34)$$

In QRL, the agent is also to learn a policy $\pi : S \times \cup_{i\in S} A_{(i)} \rightarrow [0,1]$, which will maximize the expected sum of discounted reward of each state. That is to say, the mapping from states to actions is $\pi : S \rightarrow A$, and

$$f(s) = \left|a_s^{(n)}\right\rangle = \sum_{a=00,...,0}^{\overbrace{11,...,1}^{n}} C_a |a\rangle \tag{35}$$

where probability amplitude C_a satisfies $\sum_{a=00,...,0}^{\overbrace{11,...,1}^{n}} |C_a|^2 = 1$. According to quantum parallelism, a certain unitary transformation U on the qubits can be implemented. Suppose we have such an operation which can simultaneously process these 2^m states with the TD(0) value-updating rule (**??**). It is like parallel value updating of traditional RL over all states. However, it provides an exponential-scale computation space in the m-qubit linear physical space and can speed up the solutions of related functions. In QRL, action selection is executed by measuring action $\left|a_s^{(n)}\right\rangle$ related to certain state $|s\rangle$, which will collapse to $|a\rangle$ with probability $|C_a|^2$. As the action $\left|a_s^{(n)}\right\rangle$ is the superposition 2^n possible eigenactions, finding out $|a\rangle$ usually entails interacting with changing its probability amplitude for a quantum system. The updating of probability amplitude is based on the Grover iteration, [171]. To construct the Grover iteration, the two reflections

$$\begin{aligned} U_a &= I - 2|a\rangle\langle a| \\ U_{a_0^{(n)}} &= H^{\otimes n}(2|0\rangle\langle 0| - I)H^{\otimes n} = 2\left|a_0^{(n)}\right\rangle\left\langle a_0^{(n)}\right| - I \end{aligned} \tag{36}$$

are combined. One Grover iteration is the unitary transformation $U_{Grov} = U_{a_0^{(n)}} U_a$. The procedural form of a standard QRL algorithm is, [34],

- Initialize $\left|s^{(m)}\right\rangle = \sum_{s=00,...,0}^{11,...,1} C_s |s\rangle$, $f(s) = \left|a_s^{(n)}\right\rangle = \sum_{a=00,...,0}^{11,...,1} C_a |a\rangle$ and $V(s)$ arbitrarily
- Repeat (for each episode):

  For all states $|s\rangle$ in $\left|s^{(m)}\right\rangle = \sum_{s=00,...,0}^{11,...,1} C_s |s\rangle$;

  Observe $f(s) = \left|a_s^{(n)}\right\rangle$ and get $|a\rangle$;

  Take action $|a\rangle$, observe next state $|s'\rangle$, reward r

  (a) Update state value

  $$V(s) \leftarrow V(s) + \alpha(r + \gamma V(s') - V(s))$$

  (b) Update probability amplitudes;

  Repeat U_{Grov} for L times

  $$U_{Grov}\left|a_s^{(n)}\right\rangle = U_{a_0^{(n)}} U_a \left|a_s^{(n)}\right\rangle$$

  Until for all states $|\Delta V(s)| \leq \varepsilon$

QRL achieves a good tradeoff between exploration and exploitation and can also speedup learning, [34].

5.3.3. Single-particle and Many-body Quantum System Manipulation via Q-learning

It has been shown that the performance of RL algorithms is comparable to that of optimal control methods in the task of finding short, high-fidelity protocol driving the state from an initial value to the target in nonintegrable many body quantum systems of interacting qubits. A modified version of Watkin's Q-learning algorithm, [149], with linear function approximation and eligibility traces has been used to teach the RL agent to find protocols of optimal fidelity. To manipulate the quantum system, the computer agent constructs piecewise-constant protocols of duration

T by choosing a drive protocol strength $h_x(t)$ at each time $t = j\delta t$, $j = \{0, 1, ..., T/\delta t\}$, with δt the time step size. In order to make the agent learn, it is given a reward for every protocol it constructs — the fidelity $F_h(T) = |\langle\psi\,|\psi(T)\rangle|^2$ for being in the target state after time T following the protocol $h_x(t)$ under unitary Schrödinger evolution (1). The goal of the agent is to maximize the reward in a series of attempts. Deprived of any knowledge about the underlying physical model, the agent collects information about already tried protocols based on which it constructs improved new protocols through a sophisticated biased sampling algorithm, [167]. Since in realistic applications, access to infinite control fields is not possible, the fields $h_x(t)$ can be restricted to some interval. The fidelity optimization is defined as an episodic, undiscounted RL problem. Each episode takes a fixed number of steps $N_T = T/\delta t$, where T is the total protocol duration, and δt the physical (protocol) time step. The state space $\mathcal{S} = \{s = (t, h_x(t))\}$ consists of all tuples of $(t, h_x(t))$ and corresponding magnetic field $h_x(t)$, so the RL algorithm is model-free. The action space $\mathcal{A} = \{a = \delta h_x\}$ consists of all jumps δh_x in the protocol h_x, so protocols are constructed as piecewise constant functions, e.g., bang-bang and quasi continuous protocols. The reward space $\mathcal{R} = \{r \in [0, 1]\}$ is the space of all real numbers in the interval $[0, 1]$. The rewards for the agent are given only at the end of each episode according to

$$r(t) = \begin{cases} 0, & if \quad t < T \\ F_h(T) = |\langle\psi * |\psi(T)\rangle|^2, & if \quad t = T \end{cases} \tag{37}$$

that reflects the importance of maximizing fidelity at the final time. Another essential item to set up the RL problem is the definition of environment with which the agent interacts in order to learn. The environment is defined as

$$\{i\partial_t |\psi(t)\rangle = H(t)\,|\psi(t)\rangle\,, |\psi(0)\rangle = |\psi_i\rangle\,, |\psi*\rangle\}\,. \tag{38}$$

The Hamiltonian $H[h_x(t)]$ is used to show the time dependence of the magnetic field being constructed by the agent during the episode via online Q-learning updates for specific single-particle and many-body examples. For instance, a two-level system is described as

$$H[h_x(t)] = -S^z - h_x(t)\,S^x \tag{39}$$

in which S^a are the spin-1/2 operators, and a closed chain of L-coupled qubits is considered as

$$H[h_x(t)] = -\sum_{j=1}^{L}\left[S^z_{j+1}S^z_j + gS^z_j + h_x(t)\,S^x_j\right]. \tag{40}$$

By means of the suitable protocol construction algorithm, the usefulness of Q-learning to manipulate single-particle and many-body quantum systems has been demonstrated and simulated in [167].

The balance between exploration and exploitation for learning control of quantum systems has also been addressed in a fidelity-based probabilistic Q-learning (FPQL) approach. The fulfillment of the FPQL algorithm has been tested for spin-1/2 and Λ-type atomic systems showing that FPQL algorithms achieve a better balance between exploration, and exploitation, and can also keep away from local optimal policies and speedup the learning process, [172].

6. Quantum Feedback Control

In this section, we first recall a feedback design method with an open-loop control strategy, Lyapunov control, which is specially beneficial for several complex quantum control problems. Variable structure control schemes are also another type of closed-loop control method being able to provide an alternative quantum engineering approach to manipulate quantum systems. Then, we review the two main categories of quantum feedback control methods, including measurement-based feedback methods and coherent quantum feedback methods. The former involve some kind of direct or indirect measurement on the system producing a classical output signal that is then fed to a controller in order to regognize the control input. The latter do not need the transition to classical information. The system of interest is dynamically entangled, and, therefore, shares its quantum information with another quantum system, which in turn is fed to a completely quantum network, acting as a quantum controller, [89].

6.1. Lyapunov-based Control Design

Lyapunov method has been a good strategy, especially for quantum state transformations, and provides a straightforward way to design control fields without measurement or feedback. The main object in quantum Lyapunov control theory is Lyapunov function V. Several kinds of Lyapunov functions have been studied to design an open-loop control law for quantum systems. For instance, a Lyapunov function may be selected by the distance between the actual state $|\psi\rangle$ and the desired state $|\psi_d\rangle$, [173]. Therefore, the distance decreases continuously when the first-order derivative of the Lyapunov function with respect to time is kept non-positive. It has several notions of distance depending on different ways of the transition probability, [174], but Bures distance, defined as $d_B(|\psi_1\rangle, |\psi_2\rangle) = \min_{\theta} \left\| |\psi_1\rangle - e^{i\theta} |\psi_2\rangle \right\|$, is intuitive and represents the Euclidean distance of equivalence classes. However, by considering the inconvenience in conventional processing, the Lyapunov function based on the so-called Hilbert-Schmidt distance is given by $V = \frac{1}{2}\left(1 - |\langle \psi_d|\psi\rangle|^2\right)$, which is used to prove stability. The problem is to find controls in feedback form that move the initial state to a desired target state. For simplicity and by considering practical requirements, two important assumptions are considered for the final state. First, it is an eigenstate of the unperturbed system. Second, it is not an eigenvalue of all operators H_k. Taking into account the Schrödinger equation, and by writing $\langle \psi|\psi_d\rangle$ in terms of its polar representation $\langle \psi|\psi_d\rangle = |\langle \psi|\psi_d\rangle| e^{i\angle\langle \psi|\psi_d\rangle}$ the first-order time derivative of V is

$$\dot{V} = -\frac{1}{\hbar}\sum_{k=1}^{r} u_k(t).\,\mathrm{Im}\left[\langle \psi|\psi_d\rangle \langle \psi_d|H_k|\psi\rangle\right]\cdot \tag{41}$$

A reliable design is to consider $\dot{V} \leq 0$. Accordingly, the function form of u_k can be written as

$$u_k = K_k f_k(\mathrm{Im}\left[\langle \psi|\psi_d\rangle \langle \psi_d|H_k|\psi\rangle\right]) \tag{42}$$

where $K_k > 0$ and $f_k : \mathbb{R} \to \mathbb{R}$ are functions such that $f_k(0) = 0$ and $x_k \mapsto x_k f_k(x_k)$ is positive definite for $x_k \in \mathbb{R}$. Note that when $|\psi\rangle = |\psi_d\rangle$, the control $u_k = 0$, which means the control field will disappear when the desired final state is reached, [173]. Due to the function form of u_k, it follows that $\dot{V} \leq 0$ and the system is stable. There are several lemmas for the conditions of asymptotic stability, based on these lemmas. If the coupled equation $\langle \psi_d|[H_0, H_k]|\psi\rangle = 0$, then the only solution is $|\psi\rangle = |\psi_d\rangle$, and the control system is asymptotically stable, [19].

As an example, let consider the spin 1/2 particle in σ_z representation for which the control function $u(t)$ varies the electromagnetic field in the y-direction. Therefore, the Schrödinger equation would be $i\hbar\left|\dot{\psi}(t)\right\rangle = (H_0 + u_1 H_1)|\psi(t)\rangle$ where $H_0 = \sigma_z$ and $H_1 = \sigma_y$. To perform a NOT, we have to steer the state from $|\psi_0\rangle = |1\rangle$ to $|\psi_d\rangle = |0\rangle$, (the opposite transformation is symmetrical). Since $K_1 f_1\left(\mathrm{Im}\left\langle \psi_d\left|\sigma_y\right|\psi(0)\right\rangle\right)\left\langle \psi_d\left|\sigma_y\right|\psi(0)\right\rangle \neq 0$ is satisfied, for $t > 0$, $\left\langle \psi(t)\middle|\psi_f\right\rangle \neq 0$ will hold, so the control field (42) is suitable for the system, [19]. Noting that the equation $\langle 1|-2i\sigma_x|\psi\rangle = 0$ only admits $|\psi\rangle = |1\rangle$, and the system is asymptotically stable. By choosing $f_1(x) = sgn(x)$, one has $u_1 = K_1 sgn\left\{\mathrm{Im}\left[e^{i\angle\langle \psi|\psi_d\rangle}\langle \psi_d|H_1|\psi\rangle\right]\right\}$. Since $xsgn(x) \geq 0$, the requirement of the control field is satisfied. Moreover, the control function has the form of bang-bang control as in [94].

The Hadamard gate is amongst the basic and important gates to perform quantum algorithms, and is commonly used for the transformation of a basis state into the superposition of basis states in quantum computation, [64]. Quantum Hadamard gate has been realized by choosing a proper Lyapunov function to design control fields based on Lyapunov stability theory, [175]. Considering the time evolution operator of the quantum system $U(T) = e^{-i\int_0^T H(t)dt/\hbar}$, (see also [176]), Hadamard gate can then be realized in two-step operations by choosing suitable control fields

$$\begin{aligned} U_1 &= e^{-i\sigma_y \int_0^{T_m} \Omega_1 dt} = e^{-i\sigma_y \cdot \frac{\pi}{4}} \\ U_2 &= e^{-i\sigma_x \int_0^{T-T_m} \Omega_2 dt} = e^{-i\sigma_x \cdot \frac{\pi}{2}}. \end{aligned} \tag{43}$$

Thus U_1 and U_2 can be realized as long as control fields Ω_1 and Ω_2 satisfy $\int_0^{T_m} \Omega_1(t)dt = \frac{\pi}{4}$ and $\int_{T_m}^{T} \Omega_2(t)dt = \frac{\pi}{2}$,

respectively. There exist several control field methods where the Lyapunov is adapted to time-varying and non-linear systems, and the control fields can ensure the stability of the controlled system. For this case, Lyapunov method is used to design the control fields Ω_1 and Ω_2. For quantum system $i\left|\dot{\psi}(t)\right\rangle = \tilde{H}(\varphi)\,\Omega(t)\,|\psi(t)\rangle$ with $|\psi\rangle = \begin{bmatrix} x_1 + ix_3 & x_2 + ix_4 \end{bmatrix}^T$, the equation of the real vector $x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T$ is obtained as $\dot{x}(t) = B\Omega(t)\,x(t)$, in which $B = \begin{bmatrix} \mathrm{Im}\left(\tilde{H}(\varphi)\right) & \mathrm{Re}\left(\tilde{H}(\varphi)\right) \\ -\mathrm{Re}\left(\tilde{H}(\varphi)\right) & \mathrm{Im}\left(\tilde{H}(\varphi)\right) \end{bmatrix}$. The Lyapunov function would be selected as $V(x) = \frac{1}{2}\left(x - x_f\right)^T P\left(x - x_f\right)$ where P is a positive definite symmetric matrix. Therefore,

$$\dot{V}(x) = \left(x - x_f\right)^T PB\Omega(t)\,x \tag{44}$$

where the control field is chosen as

$$\Omega(t) = -k\left(x - x_f\right)^T PBx, \quad k > 0\cdot \tag{45}$$

Let x and x_f be real vectors resulted from separating the real and imaginary parts of $|\psi\rangle$ and $\left|\psi_{T_m}\right\rangle$, respectively. Ω_1 and Ω_2 are therefore both obtained by (45) and can be used to realize U_1 and U_2, and finally the Hadamard gate, [175].

Quantum Lyapunov methods have been considerably studied and utilized. It has been found that the monotonous increasing condition of the performance index can be achieved as long as it is given as a function of expectation values of time-dependent observable operators, whose equation of motion is governed by the field-free Hamiltonian or Liouvillian system, [177]. Based on the Hilbert-Schmidt distance as a Lyapunov function, a control design for tracking natural trajectories of generic quantum states has been studied, [38, 178]. In fact, it is expected that any control system trajectory could converge to a given target state. Although LaSalle invariance in the Lyapunov methods principle ensures that any control trajectory can only converge to the invariant set, through the analysis of reference trajectory tracking for locally designed coherent quantum controls, asymptotically approaching toward any eigenstate has been achieved, [179]. Based on Lyapunov design, feedback control laws for quantum systems have been proposed, which locally asymptotically stabilize any desired eigenstate of the system Hamiltonian, [180]. Consider the finite dimensional mathematical model $i\left|\dot{\psi}(t)\right\rangle = (H_0 + H_1 u_1(t))\,|\psi(t)\rangle$, so the adapted Lyapunov function is $V = \langle\psi\,|P\,|\psi\rangle$, $P^T = P$. The set of critical points of the Lyapunov function under the constraint $\langle\psi\,|\psi\rangle = 1$ is given by the normalized eigenvectors of P. Then,

$$\dot{V} = i\,\langle\psi\,|[H_0, P]|\,\psi\rangle + i\,\langle\psi\,|[H_1, P]|\,\psi\rangle\,u_1\cdot \tag{46}$$

Since a commutator can never be sign definite, P is chosen to commute with H_0. To ensure $\dot{V} \leq 0$, the form of control field is selected as $u = -f\left(i\,\langle\psi\,|[H_1, P]|\,\psi\rangle\right)$, where f is a continuous function with $f(0) = 0$ and $xf(x) > 0, \forall x \neq 0$. Hence, the mathematical model is $i\left|\dot{\psi}(t)\right\rangle = \left(H_0 - f\left(i\,\langle\psi\,|[H_1, P]|\,\psi\rangle\right)H_1\right)|\psi(t)\rangle$, for which the set of eigenvectors of H_0 is the largest invariant set, [180].

The control strategy based on the error is widely applied to classical control systems designed by the Lyapunov method and may be utilied in quantum system. Therefore, the Lyapunov function would be defined based on the error between the initial and target states, [181]. In some cases, in order to adjust the global phase without altering the physical quantities of $|\psi\rangle$, a new real scalar control field ω corresponding to $\dot{\theta}$ is added. Hence, the control system is considered as $i\left|\dot{\psi}\right\rangle = (H(t) + \omega I)\,|\psi\rangle$, [181]. For the cases that target states are not eigenstates of the free Hamiltonian, it is possible to formulate the control problem as reference trajectory tracking, [181, 38, 179]. Take a reference trajectory $t \mapsto \left(|\psi_r(t)\rangle, u_r(t), \omega_r(t)\right)$, a smooth function of the control system $i\left|\dot{\psi}_r\right\rangle = (H_0 + u_r H_1 + \omega_r)\,|\psi_r\rangle$, and the time varying function $V(\psi, t) = \langle\psi - \psi_r\,|\psi - \psi_r\rangle$. V is positive for all $t > 0$ and all $\psi \in \mathbb{S}^{2n-1}$ and vanishes when $\psi = \psi_r$. It is easy to conclude that V is a control Lyapunov function when $|\psi\rangle$ satisfies the control system equation. The first-order time derivative of V is

$$\dot{V} = 2\,(u - u_r)\,\mathrm{Im}\left(\langle H_1\psi(t)\,|\psi_r\rangle\right) + 2\,(\omega - \omega_r)\,\mathrm{Im}\left(\langle\psi(t)\,|\psi_r\rangle\right)\cdot \tag{47}$$

Hence, the simple form of control laws to ensure $\dot{V} \leq 0$ can be selected as

$$\begin{aligned} u &= u_r(t) - a\,\mathrm{Im}\left(\langle H_1\psi(t)\,|\psi_r(t)\rangle\right) \\ \omega &= \omega_r(t) - b\,\mathrm{Im}\left(\langle\psi(t)\,|\psi_r(t)\rangle\right) \end{aligned} \tag{48}$$

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where a and b are considered positive parameters, [181]. Furthermore, several techniques based on Lyapunov have been formulated utilizing density operators for the control of a spin ensemble, [182, 183]. The Lyapunov-based feedback design does not require spins that are selectively addressable. This method makes it possible to attain control inputs also for complicated tasks such as suppressing undesired couplings in identical spin systems, [183].

6.2. Variable Structure Control

Variable structure control has been exploited as a quantum engineering approach to manipulate quantum systems, [40, 41, 42]. The method comes up with improved properties through the change of the controller structure conforming to the defined switching logic. For instance, a variable structure control scheme that can accomplish global control design has been proposed for a class of quantum systems wavefunction uncontrollable under the action of an individual controller with a fixed structure, [41]. The finite dimensional quantum control model presented in (12) can be considered as the beginning stage for controllability analysis, [75], in which A and B are typically assumed diagonal and Hermitian, respectively, and do not commute. B matrices with different structure indicate different controller structures with the matrix $B^u = u(t)\,B$ corresponding to a controller. In [41], a specific five-level system with $A = diag(1, 1.2, 1.3, 2, 2.15)$ is considered, and two quantum control models are defined as, [184],

$$\begin{cases} Model \quad I : i\hbar\dot{C}(t) = \left(A + B_1^u\right) C(t) \\ Model \quad II : i\hbar\dot{C}(t) = \left(A + B_2^u\right) C(t) \end{cases} \tag{49}$$

with corresponding controllers

$$\text{Controller I:} \quad B_1^u = \begin{pmatrix} 0 & 0 & 0 & u_1(t) & u_1(t) \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ u_1(t) & 0 & 0 & 0 & u_1(t) \\ u_1(t) & 0 & 0 & u_1(t) & 0 \end{pmatrix}$$

$$\text{Controller II:} \quad B_2^u = \begin{pmatrix} 0 & 0 & u_2(t) & 0 & 0 \\ 0 & 0 & 0 & u_2(t) & u_2(t) \\ u_2(t) & 0 & 0 & 0 & 0 \\ 0 & u_2(t) & 0 & 0 & u_2(t) \\ 0 & u_2(t) & 0 & u_2(t) & 0 \end{pmatrix}$$

The two associated graphs $G(V, E)$ (13) of models I and II are not connected. Therefore, the mentioned five-level system is wavefunction uncontrollable under the action of either controller I or II, meaning that both models are wavefunction uncontrollable. However, switching between the two controllers can make the system wavefunction controllable provided that the controller can be arbitrarily switched between controller I and controller II, [41]. As a result, the presented strategy makes a wavefunction uncontrollable system to be wavefunction controllable by involving a switching logic. However, in the bang bang control, controllers with different control values V and $-V$ can be employed as having the same controller structure. If the controller B^u can be arbitrary switched between controllers B_m^u and B_n^u $(m \neq n, 1 \leq m, n \leq M)$, the system

$$\begin{aligned} & i\hbar\dot{C}(t) = (A + B^u)\, C(t) \\ & C(t = 0) = C_0, \quad B^u \in \left\{B_1^u, \ldots, B_M^u\right\} \end{aligned} \tag{50}$$

is wavefunction controllable. It is worth mentioning that Lyapunov methods may be applicable to implement practical methods for designing the switching logic.

6.3. Quantum measurement-based Feedback Control

During the measurement-based feedback control (MFC) procedure, the system is measured in order to obtain information of the system state and, then, design the control law according to the estimation of the state. In classical

feedback control, the information of the system can simply be obtained without disturbance, while the quantum states collapse under the measurement procedure, [115]. As an example, let consider feedback process based on the outcomes of a von Neumann measurement. Each result of this measurement projects the system into the set of basis states. For each state, it is possible to perform a different unitary operation on the system. This type of measurement-based feedback have several important applications, especially in quantum error correction, [185]. Another example of quantum feedback is Maxwell's famous demon, [186], which can be usefully analysed based on von Neumann measurements, [187].

6.3.1. Continuous Quantum measurements

One possible technique to represent continuous measurements, [188, 189], is through the quantum trajectory approach, [190, 191, 192, 193]. In general, the evolution of a system state, possibly, undergoing stochastic processes, e.g. quantum jumps, is known as a quantum trajectory, [190]. This approach describes the evolution of a conditional state when a measurement record is given. Even if classical noise does not appear, such a trajectory pursues a stochastic path in time, where the randomness is a result of quantum uncertainty. These stochastic trajectories are either diffusive or jump-like in nature. The former typically appears when the observable being measured is only weakly coupled to the detector while the latter occurs when the observer's knowledge of the system state experiences a large sudden change, [191]. In addition to this approach, it is useful to estimate an unknown dynamical parameter through continuously monitoring an open quantum system. Since continuous observations of open quantum systems are naturally noisy, quantum filtering theory is useful to extract information, [194]. Quantum filtering theory is a key element for several quantum feedback control strategies, [195, 196], such as risk-sensitive optimal control, [195]. In quantum filtering, the measurement output is fed to a classical system in order to produce an estimated state of the measured quantum system. Therefore, a classical system is used to imitate a quantum system. In fact, quantum filtering acts as a bridge between a quantum system and a classical controller, [187]. In the following we will see quantum trajectories and quantum filtering approaches in more detail.

6.3.1.1 Quantum Trajectories

One way to find a quantum trajectory for a given measurement record has been the expression of quantum master equation as an average over a stochastic evolution, referred to as unravelling, for a pure quantum state, [197], which is equivalent to utilize Monte Carlo technique for simulating the master equation. For a master equation, there exists more than one stochastic equations that will unravel it, e.g., stochastic equations driven either by Wiener noise, [188], or by a point process (e.g. Poisson process), [187, 197].

(i) Quantum jumps: Let consider the unconditioned measurement-induced master equation form of (7) as

$$\dot{\rho} = -i\left[H,\rho\right] + \frac{1}{2}\sum_{\mu}\gamma_{\mu}\left(2c_{\mu}\rho c_{\mu}^{\dagger} - c_{\mu}^{\dagger}c_{\mu}\rho - \rho c_{\mu}^{\dagger}c_{\mu}\right) \tag{51}$$

in which c indicates the annihilation operator. The quantum continuous measurement represented in (51) can be written via the general description of a quantum measurement, usually referred to as a Positive-Operator-Valued-Measure (POVM), [198]. The master equation (51) driven by a point process is unraveled by the nonlinear stochastic differential equation

$$d\left|\psi_c\right\rangle = \left[-iH + \frac{1}{2}\sum_{\mu}\gamma_{\mu}\left(\left\langle c_{\mu}^{\dagger}c_{\mu}\right\rangle(t) - c_{\mu}^{\dagger}c_{\mu}\right)\right]\left|\psi_c\right\rangle dt + \sum_{\mu}\left[\frac{c_{\mu}}{\sqrt{\left\langle c_{\mu}^{\dagger}c_{\mu}\right\rangle(t)}} - 1\right]\left|\psi_c\right\rangle dN_{\mu}. \tag{52}$$

The subscript "c" shows the state conditioned on the measurement output. For each μ, N_{μ} is a point process and equals to either 1, at a set of discrete points, or 0 for the rest of the time. The stochastic Schrödinger equation (52) can alternatively be written as a stochastic master equation for the density operator, [187],

$$d\rho_c = \sum_{\mu}\mathcal{G}\left[c_{\mu}\right]\rho_c dN_{\mu}(t) + \mathcal{H}\left[-iH - \frac{1}{2}\sum_{\mu}\gamma_{\mu}c_{\mu}^{\dagger}c_{\mu}\right]\rho_c dt \tag{53}$$

where $\mathcal{G}[r]\rho = \dfrac{r\rho r^{\dagger}}{tr[r\rho r^{\dagger}]} - \rho$ and $\mathcal{H}[r]\rho = r\rho + \rho r^{\dagger} - tr\left[\left(r + r^{\dagger}\right)\rho\right]\rho$.

(ii) Quantum diffusion: Equation (51) can also be unravelled as

$$d|\psi_c\rangle = -iH|\psi_c\rangle\, dt + \sum_{\mu}\gamma_{\mu}\left(\left\langle c_{\mu}^{\dagger}\right\rangle c_{\mu} - \frac{1}{2}c_{\mu}^{\dagger}c_{\mu} - \frac{1}{2}\left\langle c_{\mu}^{\dagger}\right\rangle\left\langle c_{\mu}\right\rangle\right)|\psi_c\rangle\, dt + \sum_{\mu}\sqrt{\gamma_{\mu}}\left(c_{\mu} - \left\langle c_{\mu}\right\rangle\right)|\psi_c\rangle\, dW_{\mu} \tag{54}$$

where dW_{μ} indicate a set of mutually independent Wiener noises. The equivalent stochastic master equation is, [187],

$$d\rho_c = -i[H,\rho_c]\, dt + \sum_{\mu}\left(\gamma_{\mu}\mathcal{D}\left[c_{\mu}\right]\rho_c dt + \sqrt{\gamma_{\mu}}\mathcal{H}\left[c_{\mu}\right]\rho_c dW_{\mu}\right) \tag{55}$$

where $\mathcal{D}[c]\rho = c\rho c^{\dagger} - \frac{1}{2}c^{\dagger}c\rho - \frac{1}{2}\rho c^{\dagger}c$.

By considering the measurement strengths and efficiencies as Γ_l and η_l, respectively, the continuous measurement of quantum variables A_l for $l = 1, \ldots, m$ is, [187],

$$d\rho_c = -i[H,\rho_c]\, dt + \sum_{l=1}^{m}\left(\Gamma_l\mathcal{D}[A_l]\rho_c dt + \sqrt{\eta_l\Gamma_l}\mathcal{H}[A_l]\rho_c dW_l\right), \tag{56}$$

and the measurement output is

$$dy_l = \langle A_l\rangle\, dt + \frac{1}{\sqrt{2\eta_l\Gamma_l}}dW_l. \tag{57}$$

Continuous measurements holding Wiener noise are also referred to as weak measurement, [187]. It has been shown that weak measurements let us control the qubit better in practice than is even theoretically possible without them, [199]. The performance of a quantum control scheme taking advantage of weak measurement followed by feedback control has been studied in [50], showing that it realises the optimal recovery from noise. By taking advantage of decoherence free subspace (DFS), it has been shown that weak measurement and local feedback can be utilized for controlling entanglement generation between two qubits, [49]. Transition from weak to strong measurements has also been studied by non linear quantum feedback control, [200].

In order to propose a practical quantum feedback control scheme, the information acquisition of the system model and states is required. The estimation of the system state for feedback via projective measurements on a quantum ensemble, whose corresponding problem in classical control is well established by means of direct measurements couple with estimation procedures, is more complicated because of quantum collapse postulate, [78]. The technique for determining the quantum states of a system is named quantum state tomography in quantum information processing, [201, 202, 203]. In fact, state tomography is the process of experimentally identifying an unknown quantum state. For instance, if we are given only one copy of ρ, it would be impossible to characterize it. However, it is possible to estimate ρ if we have a large number of copies of ρ. Suppose that many copies of a single qubit density operator ρ are available. The set $I/\sqrt{2}$, $X/\sqrt{2}$, $Y/\sqrt{2}$, $Z/\sqrt{2}$, where $I = \sigma_0$, $X = \sigma_x = \sigma_1$, $Y = \sigma_y = \sigma_2$ and $Z = \sigma_z = \sigma_3$, forms an orthonormal set of matrices with reference to the Hilbert Schmidt product. Therefore, ρ can be described as

$$\rho = \frac{tr(\rho)\, I + tr(X\rho)\, X + tr(Y\rho)\, Y + tr(Z\rho)\, Z}{2}. \tag{58}$$

similarly, it is possible to estimate $tr(X\rho)$ and $tr(Y\rho)$ in order to attain an acceptable estimate for ρ. An arbitrary density operator ρ on n qubits can be expanded as

$$\rho = \sum_{v}\frac{tr(\sigma_{v1}\otimes\cdots\otimes\sigma_{vn}\rho)\,\sigma_{v1}\otimes\cdots\otimes\sigma_{vn}}{2^n} \tag{59}$$

in which the sum is over vectors $\vec{v} = (v_1, ..., v_n)$. By performing measurements, it is possible to estimate every term in the sum of (58) and attain an estimation for the density matrix, [64]. Quantum state tomography is important for quantum process tomography, [24, 64], which is the parameter identification for a quantum operation as, for instance,

the system Hamiltonian. Quantum process tomography is a procedure to completely describe the dynamics of a quantum system and can be utilizied to characterize the performance of quantum gates, quantum channels or noise processes in a system. For a d-dimensional state space system, d^2 pure quantum states are selected such that the corresponding density operators $|\psi_1\rangle\langle\psi_1|, \ldots, |\psi_{d^2}\rangle\langle\psi_{d^2}|$ construct a basis for the space of matrices. For each state, the quantum system is prepared in that state and subjected to the procedure that we intend to characterize. After the procedure has run to completion, quantum state tomography is utilized in order to determine the state output from the process $\mathcal{E}(|\psi_j\rangle\langle\psi_j|)$. The procedure of process tomography is simplified to one-qubit operation in order to provide explicit formulas useful for experimental contexts. The aim is to identify a set of operation elements E_i for the quantum process $\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger$. For simplification, it is possible to consider an equivalent of $\mathcal{E}$ by means of choosing the fixed operators $E_0 = I$, $\tilde{E}_1 = X$, $\tilde{E}_2 = -iY$ and $\tilde{E}_3 = Z$ forming a basis for the set of operators on the state space. Once the set of operators E_i has been fixed, the chi matrix representation $\mathcal{E}(\rho) = \sum_{mn} \tilde{E}_m \rho \tilde{E}_n^\dagger \mathcal{X}_{mn}$ shows that $\mathcal{E}$ can be completely described by a complex number matrix $\mathcal{X}$ with $\mathcal{X}_{mn} \equiv \sum_i e_{im} e_{in}^*$. The commutation properties of fixed operators conveniently let the $\mathcal{X}$ matrix to be determined via straightforward matrix multiplication. There are 12 parameters specified by $\mathcal{X}$ determining an arbitrary single qubit quantum operation $\mathcal{E}$. These parameters may be measured through 4 sets of experiments. Fore instance, suppose the input states are prepared as $|0\rangle$, $|1\rangle$, $|+\rangle = (|0\rangle + |1\rangle)\sqrt{2}$, and $|-\rangle = (|0\rangle + i|1\rangle)\sqrt{2}$. The following 4 matrices corresponding to $\rho'_j = \mathcal{E}(\rho_j)$ are determined by using state tomography

$$\begin{aligned}
\rho'_0 &= \mathcal{E}(|0\rangle\langle 0|) \\
\rho'_1 &= \mathcal{E}(|1\rangle\langle 1|) \\
\rho'_2 &= \mathcal{E}(|+\rangle\langle +|) - i\mathcal{E}(|-\rangle\langle -|) - (1-i)(\rho'_1 + \rho'_4)/2 \\
\rho'_3 &= \mathcal{E}(|+\rangle\langle +|) - i\mathcal{E}(|-\rangle\langle -|) - (1+i)(\rho'_1 + \rho'_4)/2
\end{aligned} \tag{60}$$

Hence, ρ'_j is known, and also $\mathcal{E}(\rho_j) = \sum_k \lambda_{jk}\rho_k$, so λ_{jk} can be determined by standard linear algebraic algorithms. We may also write $\tilde{E}_m \rho_j \tilde{E}_n^\dagger = \sum_k \beta_{jk}^{mn} \rho_k$, where β_{jk}^{mn} are complex numbers that can be determined by standard algorithms from linear algebra given operators $\tilde{E}_m$ and ρ_j. Matrix $\mathcal{X}$ is defined as $\mathcal{X}_{mn} = \sum_{jk} \kappa_{jk}^{mn} \lambda_{jk}$ where κ is the generalized inverse for matrix β. Due to the particular choice of basis and representation of the set of operators, for the above-mentioned example, matrix β can be expressed as the Kronecker product $\beta = \Lambda \otimes \Lambda$ in which $\Lambda = \frac{1}{2}\begin{bmatrix} I & X \\ X & -I \end{bmatrix}$. Hence, $\mathcal{X}$ can be expressed in terms of block matrices, [64], so

$$\mathcal{X} = \Lambda \begin{bmatrix} \rho'_1 & \rho'_2 \\ \rho'_3 & \rho'_4 \end{bmatrix} \Lambda. \tag{61}$$

In addition, an attractive approach for information acquisition about a quantum operation is quantum non demolition (QND) measurement, which has been developed in two versions: photon counting, and homodyne detection, [204, 205]. QND measurement has been proposed to evade the unwanted quantum back action in measurement, i.e., perturbation due to measurement. Let A be an observable for the system $\mathcal{S}$ with system Hamiltonian $H_{\mathcal{S}}$, and the system is measured by an apparatus $\mathcal{R}$ coupled to $\mathcal{S}$ via interaction Hamiltonian $H_{\mathcal{RS}}$ for short moments. A precise measurement of A brings the global state of $\mathcal{S}$ and $\mathcal{R}$ into $|\psi\rangle \approx \sum_i |A_i\rangle_{\mathcal{S}} |R_i\rangle_{\mathcal{R}}$, in which $|A_i\rangle_{\mathcal{S}}$, and $|R_i\rangle_{\mathcal{R}}$ indicate the eigenvectors of A corresponding to the measurement results and the corresponding states of the apparatus recording them, respectively. By denoting the Heisenberg observable model, a sequence of measurements of the observable are said to be QND measurements if and only if

$$A(t) = e^{-itH_S} A e^{+itlH_S} \quad \forall \quad t_n \quad and \quad t_m \tag{62}$$

holds when measurements are being made. If (62) holds for any choice of t_n and t_m, then A is a continuous QND variable. In the case that the condition only holds for certain discrete times, then A is stroboscopic QND variable. A is automatically QND variable if it is conserved under free evolution, i,e., $\frac{d}{dt}A(t) = -\frac{i}{\hbar}[H_S, A] = 0$. An all-electrical

QND measurement of a single electron spin in a gate-defined quantum dot has been presented in [205]. By means of feedback control of a coupled circuit QED system, a scheme has been proposed for QND measurement of Fock states of a nanomechanical resonator, [206]. It has been shown that it is possible to harness measurement back action as a form of actuation in quantum control via real-time feedback performed during a QND measurement of atomic spin-angular momentum, [207]. In addition, unconditional quantum-noise suppression in a collective spin system has been demonstrated via feedback control based on QND measurement, [208]. Overall, it is worth mentioning that any kind of methods for measuring a quantum system unavoidably disturbs the system.

6.3.1.2 Quantum Filtering

When making continuous measurement, if the dynamics of the quantum system is known, it is possible to derive an equation of motion for the full knowledge of the system determined by the continuous stream of measurement outcomes. The procedure by which the state of knowledge of a quantum system is calculated from a series of measurement results is referred to as quantum filtering, [194]. The aim is to separate the output process into some part which gives us useful information about the observable being measured and the remainder which can be discarded, [208]. Quantum filtering has been introduced as an indispensable element in a number of quantum feedback control schemes, e.g., [195]. There are three main ingredients required in order to develop quantum filtering, [209]:

- The concepts of classical probability and quantum mechanics should be captured in a non commutative probability framework.
- A non commutative generalization for the concept of conditional expectations should be provided.
- A non commutative analogue of stochastic calculus and quantum stochastic differential equations is needed to provide a broad class of models for which filtering equations can be derived, [194].

The measurement of the observable A converts a quantum probability model $(\mathcal{N}, \mathbb{P})$ - where the non commutative von Neumann algebra $\mathcal{N}$ is the set of all Hermitian operators on the Hilbert space of the system, and the function $\mathbb{P} : \mathcal{N} \rightarrow C$ is a state on $\mathcal{N}$ in which C shows the complex space - to a classical probability model $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω indicates a sample space that is the set of all elementary, mutually exclusive outcomes, the commutative σ-algebra $\mathcal{F}$ is the set of events where each event is a subset of the set Ω, and the probability measure $\mathbb{P}$ is a function associating a probability with every subset of $\mathcal{F}$: $(\mathbb{P} : \mathcal{F} \rightarrow [0, 1])$. Hence, the measurement output corresponds to a classical probability model $(\mathcal{A}, \mathbb{P})$, in which

$$\mathcal{A} = \{X : X = f(A), f : \mathcal{R} \rightarrow C\} \tag{63}$$

forms a commutative algebra. In Heisenberg model, for an arbitrary system operator $X(t) = V^{\dagger}(t) XV(t)$, where $V(t)$ is the unitary evolution operator of the total system composed of the controlled system and the input field, the following quantum stochastic differential equation holds, [210, 211],

$$dX(t) = -i[X(t), H_S]\, dt + \frac{\gamma}{2}\left\{L^{\dagger}[X(t), L] + \left[L^{\dagger}, X(t)\right] L\right\} dt + \sqrt{\gamma}\left\{dB_{in}\left[L^{\dagger}, X(t)\right] + [X(t), L]\, dB_{in}^{\dagger}\right\} \tag{64}$$

in which H_S is the free Hamiltonian of measured system, dB_{in} is the increment of the quantum Wiener process, [69], and L is one of several possible system operators. It is possible to define an output field B_{out} describing the filed leaving the system after interaction as $B_{out}(t) = V^{\dagger}(t) B_{in}(t) V(t)$ with the increment form as $dB_{out} = dB_{in} + \sqrt{\gamma} L(t)$. In case that homodyne detection is performed on $B_{out}(t)$, and also the phase of the local oscillator is set to zero, then, the operator corresponding to the output of the homodyne measurement is $dY_{out} = \frac{1}{\sqrt{\gamma}}\left(dB_{out} + dB_{out}^{\dagger}\right)$, satisfying

$$dY_{out} = \left(L + L^{\dagger}\right) dt + \frac{1}{\sqrt{\gamma}}\left(dB_{in} + dB_{in}^{\dagger}\right). \tag{65}$$

Equations (64) and (65) are defined on a quantum probability space and give the dynamics of the operators that describe the measured quantum system. Quantum filtering, [194], aims to provide an estimation $\pi(X)$ of the value of the system observable X at time t given the stream of measurement results up until that time. The event set generated by the measurement output signals up to the current time t is defined as $\mathcal{Y}_{out} = \{Y(t) | Y(t) = f[Y_{out}(\tau) : 0 \leq \tau \leq t]\}$ in which

$f(\cdot)$ is an arbitrary function. The conditional expectation of $X(t)$ on $\mathcal{Y}_{out}$ is the estimation $\pi_t(X) = \mathbb{P}(X(t)|\mathcal{Y}_{out})$, where $\mathbb{P}$ is the probability measure of $\mathcal{Y}_{out}$. The dynamical equation for $\pi_t(X)$ is called quantum filtering equation, [194],

$$d\pi_t(X) = \pi_t[\mathcal{L}(X)]\,dt + \sqrt{\gamma}\left[\pi_t\left(L^\dagger X + XL\right) - \pi_t\left(L + L^\dagger\right)\pi_t(X)\right]dW \tag{66}$$

and the corresponding output equation is

$$dY_{out} = \pi_t\left(L + L^\dagger\right)dt + \frac{1}{\sqrt{\gamma}}dW \tag{67}$$

in which dW is the increment of $W(t)$ and satisfies the classical Ito rule $(dW)^2 = dt$. Based on the information extracted by the quantum measurement, the observer's knowledge of the measured quantum system is obtained by equations (66) and (67), defined on a classical probability space. Generally, a classical stochastic system is used to imitate the dynamics of a quantum stochastic model in quantum filtering, [187, 194, 212]. An example is the problem of quantum filtering for a two-level atom driven by two counter-propagating continuous-mode photons is given in [213]. For further details on the filtering problem of many particle quantum systems refer to [214].

6.3.2. Markovian Quantum Feedback

The theory of quantum-limited feedback for the systems that are monitored continuously has been investigated in [45]. Two approaches have been considered: One based on quantum measurement and another based on Hamiltonian system-bath interactions. The former leads to a stochastic non-Markovian evolution for the density operator and the latter gives rise to a non-Markovian quantum Langevin equation. It has been shown that in the limit of negligible time delay in the feedback loop, a simple deterministic Markovian master equation is obtainable from either approaches. In Markovian quantum feedback, time delay is ignored and a memory-less controller is designed, [215]. Therefore, the equation that describes the resulting evolution is a Markovian master equation, [19, 45]. Measurement-based quantum feedback aims to utilize the output signal for engineering the dynamics of the system. By denoting equation (56) and (57) for a quantum continuous measurement of operator A, we can write the measurement and output equations

$$d\rho_c = -i[H,\rho_c]\,dt + \Gamma_A\mathcal{D}[A]\rho_c dt + \sqrt{\eta\Gamma_A}\mathcal{H}[A]\rho_c dW \tag{68}$$

$$dy = \langle A\rangle\,dt + \frac{1}{\sqrt{2\eta\Gamma_A}}dW \tag{69}$$

By introducing ρ_f as the system density operator altered by feedback control, the general form of system dynamics becomes

$$d\rho_f = \mathcal{F}[t,\{dy(\tau)\,|\tau\in[0,t]\}]\rho_c \tag{70}$$

where $\mathcal{F}[t,\{dy(\tau)\,|\tau\in[0,t]\}]$ indicates a super operator depending on the output signal for all past times. For most studies, quantum feedback control is presented through varying the parameters in the system Hamiltonian based on the output signals. For instance, in case of Markovian quantum feedback, the control is applied by adding a Hamiltonian term proportional to the measured signal $dy(t)/dt$ generating the dynamics

$$d\rho_f = (\exp(\mathcal{K}dy) - 1)\rho_c \tag{71}$$

in which $\mathcal{K}$ is a super operator defined by $\mathcal{K}\rho_c = -i[F,\rho_c]$ for some Hermitian operator F. Combining equations (68) and (71), the modified closed-loop stochastic master equation would be obtained as, [216, 187],

$$d\rho_f = \left\{-i\left[H,\rho_f\right] + \Gamma_A\mathcal{D}[A]\rho_f\right\}dt - i\left[F, A\rho_f + \rho_f A\right]dt + \frac{1}{\eta}\mathcal{D}[F]\rho_f dt + \mathcal{H}\left[\sqrt{\eta}A - \frac{i}{\sqrt{\eta}}F\right]\rho_f dW \tag{72}$$

which results in the Wiseman-Milburn, [216, 45, 187, 217], equation by averaging over the Wiener noise, so

$$\dot{\rho} = -i[H,\rho] + \Gamma_A\mathcal{D}(A)\rho - i[F, A\rho + \rho A] + \frac{1}{\eta}\mathcal{D}[F]\rho \tag{73}$$

in which $\rho = E(\rho_f)$. The two last terms in equation (73) are induced by feedback. The term $-i\left[F, A\rho + \rho A\right]$ steers the system dynamics to desired effects, while $\frac{1}{\eta}\mathcal{D}\left[F\right]\rho$ indicates decoherence effects of feedback. Equation (73) is the simplest measurement-based quantum feedback approach and can be utilized to resolve several problems through appropriate choices for operators A and F. This equation can also be written in the Lindblad form, [216, 45, 187],

$$\dot{\rho} = -i\left[H + \frac{(AF + FA)}{2}, \rho\right] + \mathcal{D}\left[A - iF\right]\rho + \frac{1-\eta}{\eta}\mathcal{D}\left[F\right]\rho\cdot \tag{74}$$

In Markovian quantum feedback, the feedback Hamiltonian is usually chosen as a linear function of feedback signal. For instance in [217], the feedback Hamiltonian for the Markovian feedback of the homodyne photocurrent is considered as $H_{fb}(t) = F(t) \times I(t)$, where $I(t)$ is the singular and of indefinite sign current from measurement. Feedback Hamiltonian enters the system Hamiltonian to change the evolution of the system. Quantum Markovian feedback has been used in several aspects of physical problems, such as pure state stabilization, [218], manipulating quantum entanglement, [219, 220], and spin squeezing [221]. In addition to pure state stabilization, the potential of output-feedback Markovian control schemes for noiseless-subspace generation has been addressed in [222]. Especially, explicit outcomes for the synthesis of stabilizing semigroups and noiseless subspaces in finite-dimensional Markovian systems are attained in [222]. Moreover, it has been shown that quantum feedback control can be utilized as a quantum error correction procedure, [223]. In the case that feedback delay cannot be neglected, the feedback Hamiltonian must include a delay parameter. The effect of the time delay is qualitatively similar to that of inefficient detection. In [224], the problem of measurement-based feedback control for quantum systems has been studied, and the time to compute the filter-based control input is considered through the input delay, [224]. In addition, the effects of time delay in feedback has been investigated for the control of linear quantum systems, [225], with an emphasis on the relation between the potentially achievable control performance and time delay.

In spite of its simplicity, Markovian quantum feedback has some limitations. Feeding back the measurement signal at every instant of time decreases the optimal use of extracted information from measurement. Hence, the measurement results, given in (57), should be processed via stochastic Mater equation. By knowing ρ_c at each time, $\langle A\rangle = tr(A\rho_c)$ is also known, so the first term of (57) does not provide any new information about the system. Hence, the state of knowledge is modified by the second term of the equation, including noise dW. As the measurement proceeds, ρ_c tends to some pure state $|\psi\rangle$, and $\langle A\rangle$ tends to the true mean value $\bar{A} = \langle\psi|A|\psi\rangle$, so if A is a Hermitian observable, and $|\psi\rangle$ is an eigenstate of both the system Hamiltonian and A, the system remains in the state $|\psi\rangle$, under the condition that the system is not subject to other noise sources. In this case, $\bar{A}$ is constant, and it is possible to obtain an estimate of $\bar{A}$ of ever increasing accuracy by averaging the obtained measurement results divided by time, [226],

$$Y_A(t) = \frac{1}{t}\int_0^t dy = \frac{1}{t}\int_0^t \langle A\rangle\, dt' + \frac{1}{t\sqrt{2\eta\Gamma_A}}\int_0^t dW\cdot \tag{75}$$

Hence, as the time goes to infinity, the average of the noise tends to zero so that $Y_A(t) \to \bar{A}$. However, $\bar{A}$ is usually not constant for systems to be controlled, but time averaging can still be used to estimate $\bar{A}$. To do so, we should obtain the average of signal over a time period T long enough to reduce the noise dW, and short enough to avoid tangible changes on $\bar{A}$. A weighting factor, for instance an exponential weighting function as used for the following equation, can be included in order to cut down the dependence on the estimate of $\bar{A}$ on the measurement results that are too far in the past, [187],

$$\tilde{Y}_A(t) = \frac{1}{T}\int_{t-T}^{t} \exp\left(-\gamma_f t'\right)\left(\langle A\rangle\, dt' + \frac{1}{\sqrt{2\eta\Gamma_A}}dW\right) \tag{76}$$

When $T \ll 1/\gamma_f$, the estimate converges as, [187],

$$\tilde{Y}_A(t) - \bar{A}(t) = \exp\left(-\gamma_f t\right)\left[\tilde{Y}_A(0) - \bar{A}(0)\right]\cdot \tag{77}$$

Such filtering equations have been used in quantum feedback control in circuit QED, [227, 228]. In [227], a filtering equation changes the state evolution to contain the back action of the measurements, while simultaneously enforcing

consistency with the observed measurement outcomes. In the solid-state circuit QED system and based on the homodyne measurement in dispersive regime, it has been shown that a homodyne-current-based feedback is able to create and stabilize highly entangled two-qubit Bell states in the presence of moderate noisy environment, [228]. Moreover, an extended analysis for the current based Markovian feedback leading to an improved filtered-current-based feedback scheme has been presented. The work has also been extended to three-qubit GHZ states, [229].

6.3.3. Bayesian Quantum Feedback

In Bayesian quantum feedback (also called state-based feedback), [216], the procedure is organized in two stages. In the first step, the system state is estimated, and the best estimations of the dynamical variables are attained continuously from the measurement record. In the second step, the estimates are fed back in order to control the dynamics of the system, [230, 231]. In [231], some arbitrary functional of the whole history of the measurement outcomes is considered to modify the evolution of the system. The resulting dynamics of the system is, thus, non-Markovian, while the system dynamics and controller remain Markovian. This situation is similar to the one in classical control theory. To exploit the information provided by measurement, it is necessary to process the measurement outcomes through the stochastic master equation (56) in order to obtain ρ_c, which, together with the knowledge of the system dynamics, discover the probabilities of the outcomes of any measurement on the system at any time in future. Hence, any optimal scheme to control the system can be defined as a law to choose Hamiltonian at time t as $H(t) = f(\rho_c(t), t)$. Resources required to solve the stochastic master equation in real time are prohibitive. However, it is possible to cut down the computational overhead by selecting an initial estimate for ρ_c containing only a small number of parameters, so the stochastic master equation reduces to a stochastic differential equation for these parameters. Hence, taking the integration of a partial differential equation is simplified to integrating a set of ordinary differential equations, which are coupled [232]. Since the source of decoherence can be described according to the emission and absorption of phonon, [233], the ansatz is considered to be a mixture of symmetric and antisymmetric superpositions of pure Gaussian wavepackets, and the full ansatz for the state of a resonator is considered as $\rho = P|+\rangle\langle+| + (1-P)|-\rangle\langle-|$, where P indicates the mixing probability and $|\pm\rangle = (|G_+\rangle \pm |G_-\rangle) / \sqrt{2(1 \pm \chi)}$ in which $|G_\pm\rangle$ is the pair of general pure Gaussian states with centroids $(\pm\bar{x}, \pm\bar{p})$, variances V_p and V_x, and symmetrized covariance C, and $\chi = \mathrm{Re}[\langle G_+|G_-\rangle]$ demonstrates the overlap between the two Gaussians. Hence, the ansatz contains six parameters including $\bar{x}$, $\bar{p}$, V_p, V_x, C and P, [233]. For linear quantum systems, an ansatz with a small number of parameters provides an exact solution to the stochastic master equation. By assuming that the initial states of the noise-driven linear systems are Gaussian, then the states of the system will stay Gaussian during the evolution, and the dynamics of these variables are precisely regenerated via those of a classical linear system driven by Gaussian noise, and subjected to continuous measurements of the same observable. To reproduce the quantum dynamics correctly, for every continuous measurements made on the system, one needs to add a noise source to the classical system in order to imitate Heisenberg's uncertainty principle, [187]. Obtaining linear dynamics requires observables with unbounded spectrum, e.g., the position $\hat{q}$ of a particle has $\lambda(\hat{q}) = \mathbb{R}$. Consider a system with N such position observables that all commute. In order to obtain a complete set of such observables, it is necessary to include a canonically conjugate momentum $\hat{p}_m$ for each particle position $\hat{q}_m$, where the canonically conjugate pair obeys the commutation relation $[\hat{q}_m, \hat{p}_m] = i\hbar$. In order to connect with the classical theory, the complete set of observables is defined as the vector, [216],

$$x = (\hat{q}_1, \hat{p}_1, ..., \hat{q}_N, \hat{p}_N)^T \tag{78}$$

The satisfied commutation relations can be written down as

$$[x_n, x_m] = i\hbar\Sigma_{nm} \tag{79}$$

where Σ indicates the $(2N) \times (2N)$ symplectic matrix, [216, 234],

$$\Sigma = \bigoplus_{n=1}^{N} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \Sigma^* = -\Sigma^T = -\Sigma^{-1}. \tag{80}$$

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For a system with such a phase-space structure, it is possible to define a Gaussian state determined by its mean vector $\langle x \rangle$ and covariance matrix C of the position and momentum variables with the (m, n)-element expressed as

$$C_{n,m} = C_{m,n} = \langle \Delta x_n \Delta x_m + \Delta x_m \Delta x_n \rangle / 2 \cdot \tag{81}$$

For such states, the Schrödinger–Heisenberg uncertainty relation is a necessary and sufficient condition on C to describe a valid quantum state, written as the linear matrix inequality (LMI), [235],

$$C + i\hbar\Sigma/2 \geq 0 \cdot \tag{82}$$

This LMI can be obtained from equations (79) and (81), because $C + i\hbar\Sigma/2 = \left\langle (\Delta x)(\Delta x)^T \right\rangle$. The system Hamiltonian is quadratic in linear systems, and the dissipation operator L is linear in x, so

$$H_S = \frac{1}{2} x^T G x - x^T \Sigma b u(t), \quad L = l^T x \tag{83}$$

in which matrix G is real and symmetric, and the vectors b and l are real and complex, respectively. The second term of H in (83) describes the applied force by feedback controller, [187]. From the quantum stochastic differential equation (64), the dynamics of the controlled system can be defined by

$$dx = Axdt + budt + i\sqrt{\gamma}\Sigma \left[l dB_{in}^{\dagger} - l^* dB_{in} \right] \tag{84}$$

in which matrix $A = \Sigma \left[G + \mathrm{Im}\left(\hat{l}^* \hat{l}^T \right) \right]$. From (65), the output equation is expressed as

$$\begin{aligned} dY_{out} &= Fxdt + \frac{1}{\sqrt{\gamma}} \left(dB_{in} + dB_{in}^{\dagger} \right), \\ F &= l^T + l^{\dagger} \cdot \end{aligned} \tag{85}$$

After measurement, the dynamics of the linear quantum system can be completely represented through the conditional mean and covariance. The best estimation of the system observable, $\pi_t(x)$, obeys the filter equation

$$d\pi_t(x) = A\pi_t(x)\, dt + bu_t dt + \left[\frac{1}{\hbar} C_{c,t} F^T + \Sigma^T \,\mathrm{Im}(l) \right] \times \left[dY_t - F\pi_t(x)\, dt \right] \tag{86}$$

where $C_{c,t}$ expresses the symmetric covariance matrix as

$$C_{c,t} = \mathbb{P}\left(C_t \left| \mathcal{Y}_{out,t} \right. \right) = \begin{bmatrix} \Delta\hat{q}_t^2 & \frac{1}{2}\left(\Delta\hat{q}_t \Delta\hat{p}_t + \Delta\hat{p}_t \Delta\hat{q}_t \right) \\ \frac{1}{2}\left(\Delta\hat{q}_t \Delta\hat{p}_t + \Delta\hat{p}_t \Delta\hat{q}_t \right) & \Delta\hat{p}_t^2 \end{bmatrix} \tag{87}$$

in which $\Delta\hat{q}_t = \hat{q}_t - \pi_t(\hat{q})$ and $\Delta\hat{p}_t = \hat{p}_t - \pi_t(\hat{p})$. The covariance matrix C_c modifies in time based on the deterministic Riccati differential equation

$$\begin{aligned} \dot{C}_{c,t} &= AC_{c,t} + C_{c,t}A^T + D - \frac{1}{\hbar}\left[C_{c,t}F^T + \hbar\Sigma^T \,\mathrm{Im}(l) \right] \times \left[FC_{c,t} + \mathrm{Im}\left(l^T \right)\Sigma \right] \\ C_{c,0} &= \mathbb{P}\left(C_0 \left| \mathcal{Y}_{out,0} \right. \right) \end{aligned} \tag{88}$$

where $\hbar D = \Sigma\, \mathrm{Re}\left(l^* l^T \right) \Sigma^T$. Accordingly, the optimal filter for the linear quantum system (84) is expressed by the closed set of equations (86) and (88), called the quantum Kalman filter, [230, 231, 210, 236]. For the linear quantum system driven by the quantum Gaussian noise, one needs to find an optimal control input u^{opt} which is a function of the observations $Y_s (0 \leq s \leq t)$ in a way that the quadratic cost function of the system state x

$$J[u_t] = \left\langle \frac{1}{2} x_T^T S\, x_T + \frac{1}{2} \int_0^T \left[x_\tau^T Q x_\tau + u_\tau^T R u_\tau \right] d\tau \right\rangle \tag{89}$$

is optimized. The matrices M, N, and R reflect requirements for the the control strategy. Consider $M \geq 0$, $N \geq 0$, and R to be a positive scalar number r. Hence, if we are restricted in the magnitude of the control input, a large value for r must be selected. This problem can be solved via dynamical programming method, [237]. The optimal control input is given by

$$u_t^{opt} = -(2/r)\, b^T K_t \pi_t(x) \tag{90}$$

where the real symmetric matrix K_t is the solution to the Riccati differential equation

$$\begin{aligned} &\dot{K}_t + K_t A + A^T K_t - \frac{2}{r} K_t b b^T K_t + \frac{1}{2} M = O, \\ &K_T = N\cdot \end{aligned} \tag{91}$$

It can be seen that u_t^{opt} depends only on the solution to equations (86) and (88) at time t. A controller satisfying this desirable property is called a separated controller. Separation theorem can be applied for linear systems with quadratic cost functional and Gaussian noises. The values of optimal estimations are fed back, and, when computing the feedback needed for the optimal control, one may suppose that the dynamical variables are precisely known, [78, 230]. To see the general discussion on this, refer to [238].

In quantum LQG control problems, the optimal control is a linear feedback control law as well. However, the controller can be either a classical, [53], or a quantum controller, [52]. In case of classical controllers, classical LQR control results can be applied for quantum LQR problems, [23, 230, 53], while for quantum controllers, it is necessary to add some extra constraints on coefficient matrices in order to make sure that the controller is physically realizable, [51]. The additional constraints increase the complexity considerably, and, so numerical algorithms - e.g., a numerical process according to an alternating projection algorithm, [52] - are required to solve such problems. Under several reasonable assumptions (e.g., n_y is even, $n_w \geq n_y$ and $F_w = I + idiag(J, \ldots, J)$, [51]), a system of the form (11) is physically realizable if and only if:

$$iA\Theta + i\Theta A^T + BT_w B^T = 0 \tag{92}$$

$$B\begin{bmatrix} I_{n_y \times n_y} \\ 0_{(n_w - n_y)\times n_y} \end{bmatrix} = \Theta C^T diag_{N_y}(J) \tag{93}$$

$$D = \begin{bmatrix} I_{n_y \times n_y} & 0_{n_y \times (n_w - n_y)} \end{bmatrix} \tag{94}$$

where $N_y = \dfrac{n_y}{2}$ and $T_w = \dfrac{1}{2}\left(F_w - F_w^T\right)$, [78, 51]. For more details on physical realizability see [51].

6.4. Quantum Coherent Feedback Control

The concept of coherent quantum feedback was first presented in [239], and it was experimentally shown through NMR quantum information processing methods, [240]. It has also previously been presented in quantum optics as the all-optical feedback (compared to electro-optical feedback control), [241]. In this method, the feedback controller is a quantum system, and the control operations are made up of unitary transformations. This is different from measurement-based methods, where the feedback information and feedback controller are both classical. Since coherent feedback control uses full quantum information, it is able to do a number of tasks that are not possible to be performed via controllers using a classical information feedback loop, [239]. Coherent control does not involve measurement, and, thus, that the measurement noise is eliminated while both the controller, and the plant are coherently connected quantum systems. Next, the concept of direct coherent feedback control is explained for quantum state to state transfer. In this method, a unitary interaction is used between the system and controller. Afterwards, following the concept of all optical feedback, quantum field mediated feedback control for which the interaction is mediated by travelling wave fields will be discussed. Then, it is demonstrated how the series and concatenation products let describe quantum feedback control transparently.

6.4.1. Quantum State Transformation

By means of coherent feedback method, it is possible to transfer an unknown initial quantum state to a desired target state without destroying it. For instance, consider the state transformation from an unknown initial state $|\psi_0\rangle =$

$\alpha|0\rangle+\beta|1\rangle$ to the desired final state $|1\rangle$. In case the initial state is fully unknown, the state of the qubit is considered as the density matrix $\rho = (1/2)|0\rangle\langle 0| + (1/2)|1\rangle\langle 1|$. It is not possible to do this transformation via a unitary operation on the system, since the initial state must be known in order to choose the appropriate unitary. In order to utilize a unitary interaction for preparation of the system state in $|1\rangle$, it is necessary to have the controller in a pure state, for instance starting from $|0\rangle$, then an interaction is turned on to transform the controller to the state $|1\rangle$ provided that the qubit under control is in state $|1\rangle$. In order to correlate $|\psi_0\rangle$ with the controller qubit, which is considered to start initially from the state $|0\rangle$, we need to apply a unitary such that $U(|\psi_0\rangle \otimes |0\rangle) = \alpha|0\rangle \otimes |0\rangle + \beta|1\rangle \otimes |1\rangle$, where the unitary can be written as

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{95}$$

As seen, the controller is in state $|1\rangle$ if and only if the qubit under control is in state $|1\rangle$. Therefore, it is necessary to transfer the state of controlled qubit from $|0\rangle$ to $|1\rangle$ only if the state of the controller is $|0\rangle$, which can be done through the feedback unitary

$$U_{fb} = (|0\rangle\langle 1| + |1\rangle\langle 0|) \otimes |0\rangle\langle 0| + I \otimes |1\rangle\langle 1|. \tag{96}$$

The feedback unitary may be applied to the joint two-qubit system to produce the final state and complete the feedback procedure in a way that the controlled qubit take the state $|1\rangle$ for any value of α and β, so

$$U_{fb}(\alpha|0\rangle \otimes |0\rangle + \beta|1\rangle \otimes |1\rangle) = |1\rangle \otimes |\psi_0\rangle. \tag{97}$$

As it can be seen, $|\psi_0\rangle$ is not destroyed, and the quantum information in the state of the controlled qubit at any time is transferred to the controller. In coherent feedback method, the controller performs the similar action as a measurement that projects the system state onto the basis $\{|0\rangle, |1\rangle\}$. However, since the projection is not really performed, and the overall procedure is coherent, the quantum information carried by the controlled qubit would not be destroyed. In fact, the controller becomes correlated with the system in the basis $\{|0\rangle, |1\rangle\}$ and performs an action according to the state of the system, [187]. Feedback control by means of quantum controllers is not stochastic. In particular, the target state to which the quantum controller transfers the system is entangled with another quantum system. Entanglement is a non-local quantum phenomenon that cannot be generated by classical controllers, [242]. In this method, often termed as "direct coherent feedback", a unitary (Hamiltonian) interaction has been employed between the system and controller, and the Hamiltonian description of the joint system does not demonstrate a loop, but it presents an interaction between the two systems.

6.4.2. All Optical Feedback Control

In the previous section, it was explained that a quantum controller can perform feedback even if the system description does not require an explicit loop. However, there exists a technique to make the loop explicit for a quantum controller where there is no measurement in a sense that the system is coupled to a travelling wave optical field that propagates from the system to the controller and closes the loop via another travelling wave field from the controller to the system. To do so, the two travelling wave fields must continue propagating after their interaction with the systems. This type of coherent feedback is referred to as "field-mediated feedback". For instance, instead of detection of the emitted light in electro-optic feedback, the light can be reflected around a loop back into the source cavity or, more effectively, to another cavity coupled to the source cavity in some way. In [241], two optical cavities are considered in horizontal (cavity 1), and vertical (cavity 2) states coupled directly by a nonlinear crystal. The output beam from cavity 1 is fed through a combination of a polarization beam splitter and a Faraday rotator, and, then into cavity 2. The combination of the polarization-sensitive beam splitters and Faraday rotator forms an optical isolator, [243]. Therefore, the output field from cavity 1 travels in a unidirectional way to cavity 2, so such mechanism is termed as a cascade coupling between the cavities, [241]. For such feedback scheme, cavity 1 is the system under control, and cavity 2 is the controller, which takes the system information from the one-way field and applies feedback through the direct coupling, [187, 241]. An optical cavity is normally described by the Hamiltonian

$$H_{tot} = H_{\mathcal{S}} + H_{\mathcal{B}} + H_{\mathcal{I}} \tag{98}$$

where H_S is a function of internal-mode operators only, H_B indicates the internal Hamiltonian of the bath, and H_I expresses the interaction between bath and cavity field, which is taken to be linear. The behaviour of the internal mode or modes may be either calculated by master equation or quantum Langevin equations. In [241], three kinds of interaction Hamiltonians are considered in which c_2 is the annihilation operator of the driven cavity (cavity 2), K is an arbitrary observable of the mode on the source cavity (cavity 1), and B is an arbitrary operator:

- $H_I = \hbar c_2^\dagger c_2 K$ realized via a χ^2 nonlinear crystal and can be utilized to regenerate feedback by means of photon counting.
- $H_I = \hbar\left(c_2 + c_2^\dagger\right) K$ realized via a χ^2 nonlinear crystal and an extra filed that may be treated classically.
- $H_I = \hbar\left(c_2 B^\dagger + c_2^\dagger B\right)$ provides feedback with no measurement-based counterpart and is able to generate non classical states in system mode.

Regarding the last interaction Hamiltonian, if $B = i\lambda\left(c_1 + \mu c_1^\dagger\right)$ with $\lambda \in \mathbb{R}^+$ and $\mu \in \mathbb{R}$, then, the resulting feedback produces squeezed states of the system mode for $-1 < \mu < 1$. This can be realized experimentally by combining mode conversion via a polarization rotator using a χ^2 nonlinear crystal.

6.4.3. Quantum Network Modelling Methodology

In the previous section, a one-way connection was considered from the system to the controller, while the coupling for the feedback part of the loop was not unidirectional. However, it is possible to use a cascade connection for both so that we have a complete unidirectional loop. General and efficient methods have been recognized in order to describe networks of quantum components and have been emerged in quantum optics, quantum computing and information processing, [210, 64, 211, 244]. For instance, a general input-output theory for quantum dissipative systems is developed with the possibility of relating output to input via the internal dynamics of a system, [211, 244], such that systems can easily be connected together by setting the input of one system equal to the output of another. In these formulations, every system is expressed by a Hamiltonian together with the operators through which it is coupled to the input/output fields, where the fields can be coupled together by means of a beam splitter. With the concept of quantum networks, [245], it is possible to describe this formalism to combine circuit elements to networks where each unit is described by the parameter list $G = (S, L, H)$ due to the parametrization of Hudson-Parthasarathy, [246], Collett-Gardiner, [211, 244], and close relation to the standard form of Lindblad generator, that will be explained in (101). The scattering matrix $S_{n\times n}$ is square with operator entries such that $S^\dagger S = S S^\dagger = I$. The coupling vector $L_{n\times 1}$ is a column vector with operator entries through which the system couples to the inputs, and H is a self adjoint operator indicating the internal Hamiltonian of the system. These parameters specify an open quantum system coupled to field channels with corresponding gauge processes

$$\begin{aligned} B(t) &= \begin{pmatrix} B_1(t) \\ \vdots \\ B_n(t) \end{pmatrix}, \\ \Lambda(t) &= \begin{pmatrix} B_{11}(t) & \cdots & B_{1n}(t) \\ \vdots & \vdots & \vdots \\ B_{n1}(t) & \cdots & B_{nn}(t) \end{pmatrix} \end{aligned} \tag{99}$$

where the vector $B(t)$ and the matrix $\Lambda(t)$ are quantum Wiener processes and quantum Poisson processes, respectively. The noise processes are integrals of the input fields denoted by $b_i(t), (i = 1, \ldots, n)$, which construct the vector of inputs to the system $b_{in}(t) = [b_1(t), \cdots, b_n(t)]^T$. Hence,

$$\begin{aligned} B_i(t) &= \int_0^t b_i(\tau)\, d\tau, \\ B_{ij}(t) &= \int_0^t b_i^\dagger(\tau)\, b_j(\tau)\, d\tau. \end{aligned} \tag{100}$$

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All differentials are understood in the Ito sense, [246]. We assume that all processes are canonical, meaning that the following non vanishing second order Ito products exist:

$$\begin{aligned} &dB_j(t)\,dB_k(t)^\dagger = \delta_{jk}dt \\ &dB_{jk}(t)\,dB_l(t)^\dagger = \delta_{kl}dB_j(t)^\dagger \\ &dB_j(t)\,dB_{kl}(t) = \delta_{jk}dB_l(t) \\ &dB_{jk}(t)\,dB_{lm}(t) = \delta_{kl}dB_{jm}(t)\cdot \end{aligned} \tag{101}$$

For the open quantum system $G = (S, L, H)$ with canonical inputs, the unitary motion of the system is determined by

$$dV(t) = (tr\left((S-I)\,d\Lambda^T\right) + dB^\dagger L - L^\dagger S\,dB - \frac{1}{2}L^\dagger L dt - iHdt)V(t) \equiv dG(t)\,V(t) \tag{102}$$

with initial condition $V(0) = I$. Equation (102) demonstrates the definition of the generator $dG(t)$. The operators constituting the above mentioned system parametrization are supposed to be defined on an underlying Hilbert space, referred to as the initial space. In the Heisenberg model, a system operator $X(t) = V(t)^\dagger XV(t)$ defined on the initial space satisfies the quantum stochastic differential equation

$$\begin{aligned} dX(t) = &\left(\mathcal{L}_{L(t)}(X(t)) - i\left[X(t), H(t)\right]\right)dt + dB^\dagger(t)\,S^\dagger(t)\left[X(t), L(t)\right] \\ &+ \left[L^\dagger(t), X(t)\right]S(t)\,dB(t) + tr\left(\left(S^\dagger(t)\,X(t)\,S(t) - X(t)\right)d\Lambda^T(t)\right) \end{aligned} \tag{103}$$

in which $\mathcal{L}_{L(\cdot)}$ indicates the Liouville super operator defined by

$$\mathcal{L}_L(X) = \frac{1}{2}L^\dagger[X, L] + \frac{1}{2}\left[L^\dagger, X\right]L = \sum_{j=1}^{n}\left(\frac{1}{2}L_j^\dagger\left[X, L_j\right] + \frac{1}{2}\left[L_j^\dagger, X\right]L_j\right). \tag{104}$$

Since the dynamics is unitary, the commutation relations are preserved. The output fields corresponding to the inputs $B(t)$ and $\Lambda(t)$ are given by

$$\begin{aligned} &B_{out}(t) = V^\dagger(t)\,B(t)\,V(t), \\ &\Lambda_{out}(t) = V^\dagger(t)\,\Lambda(t)\,V(t)\cdot \end{aligned} \tag{105}$$

They are related to the system inputs through the quantum stochastic differential equations

$$\begin{aligned} &dB_{out}(t) = S(t)\,dB(t) + L(t)\,dt \\ &d\Lambda_{out}(t) = S^*(t)\,d\Lambda(t)\,S^T(t) + L^*(t)\,L^T(t)\,dt + L^*(t)\,dB^T(t)\,S^T(t) + S^*(t)\,dB^*(t)\,L^T(t) \end{aligned} \tag{106}$$

satisfying the quantum Ito rules for quantum stochastic calculus in (98). The dynamics of a system coupled to input fields is given by the quantum Langevin equations, which can be solved directly for linear quantum systems. In order to perform calculations for nonlinear quantum systems, one must transform the Heisenberg equations of the input-output formalism to master equations (7), in which $\mathcal{D}(\rho) = \sum_j\left(L_j\rho L_j^\dagger - \frac{1}{2}L_j^\dagger L_j\rho - \frac{1}{2}\rho L_j^\dagger L_j\right)$. Although the master equation does not depend on the scattering matrix S, this matrix plays an important role in describing the architecture of the input channels as in (103). Moreover, it affects more complex quantum input-output systems, as it is the case of cascade systems. In order to configure and describe a quantum network, given the systems $G_1 = (S_1, L_1, H_1)$ and $G_2 = (S_2, L_2, H_2)$, the following two powerful tools are needed, [245]:

- Concatenation product

$$G_1 \boxplus G_2 = \left(\begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}, \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}, H_1 + H_2\right). \tag{107}$$

- Series product

$$G_2 \triangleleft G_1 = \left(S_2S_1, L_2 + S_2L_1, H_1 + H_2 + \frac{1}{2i}L_2^\dagger S_2 L_1 - \frac{1}{2i}L_1^\dagger S_2^\dagger L_2\right). \tag{108}$$

Let consider the output B_{out_1}, which results from interaction with the internal cavity mode a, is fed back as the input B_2. The coupling operators for the two field channels are modelled as $L_1 = L_2 = \sqrt{\gamma}a$, where γ indicates the damping rate. Before feedback, the cavity is expressed as

$$G = \left(I, \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}, 0\right) = (1, L_1, 0) \boxplus (1, L_2, 0) \cdot \tag{109}$$

In addition, the phase shift in the loop is described by the unit $(S, 0, 0)$, where $S = e^{i\theta}$. Therefore, the closed-loop system is described by

$$G_{cl} = (1, L_2, 0) \triangleleft (S, 0, 0) \triangleleft (1, L_1, 0) = \left(S, S L_1 + L_2, \frac{1}{2i}\left(L_2^* S L_1 - L_1^* S^* L_2\right)\right) \tag{110}$$

so $G_{cl} = (S_{cl}, L_{cl}, H_{cl})$ describes the overall closed-loop feedback, in which

$$\begin{aligned} S_{cl} &= S \equiv e^{i\theta} \\ L_{cl} &= S L_1 + L_2 \equiv \left(1 + e^{i\theta}\right)\sqrt{\gamma}a \\ H_{cl} &= \mathrm{Im}\{L_2^* S L_1\} \equiv \gamma \sin\theta a^\dagger a \end{aligned} \tag{111}$$

that entails the following Heisenberg dynamical equation for the cavity mode

$$da = -\left[a, \left(1 + e^{i\theta}\right)\sqrt{\gamma}a^\dagger\right] dB_1 - \frac{\gamma}{2}\left(1 + e^{i\theta}\right)\left(1 + e^{-i\theta}\right) a dt - i\gamma \sin\theta a dt \equiv -\left(1 + e^{i\theta}\right)\left(\sqrt{\gamma} dB_1 + \gamma a dt\right) \tag{112}$$

and the input/output relation is

$$dB_{out_2} = e^{i\theta} dB_1 + \left(1 + e^{i\theta}\right)\sqrt{\gamma} a dt \tag{113}$$

where dB_1 is the input field to source cavity, and dB_{out_2} is the output field from the controller cavity which leaves the unidirectional feedback loop. To see more details, and also the application of quantum network methodology for measurement-based feedback, refer to [245, 247].

6.4.4. Quantum Transfer Function Modelling Methodology

As seen, the algebraic theory behind quantum feedback networks has been expanded for open quantum dynamical systems. However, for the particular case of linear systems with instant feedback relations, it is possible to deduce the transfer functions in complete agreement with the mathematical rules attained in the nonlinear case. If we specify the parametrization used in [245], such that all the systems are linear, and consider the representation of their motion dynamics in the frequency space, then the framework for quantum transferormation function methods are obtained, [187]. For linear quantum systems, these can be derived in series, cascade, and feedback arrangements mediated by splitter devices, [248]. Let consider a quantum mechanical system consisting of a family of harmonic oscillators with annihilation operators $\left\{a_j : j = 1, \cdots, m\right\}$, $a = (a_1, \cdots, a_m)^T$ with the canonical commutation relations

$$\left[a_j, a_k\right] = \left[a_j^\dagger, a_k^\dagger\right] = 0 \cdot \tag{114}$$

Since the common linear quantum network represented by $G = (S, L, H)$ is considered, the below simplifying assumptions are satisfied:

- The scattering matrix S is a matrix of numbers instead of operators, so S_{jk} are scalars.
- The field coupling operators are linear mixtures of the annihilation operators, i.e., there exist c_{jk} such that $L_j \equiv \sum_k c_{jk} a_k$.
- The system Hamiltonian H is quadratic, meaning that there is w_{jk} in a way that $H = \sum_{jk} a_j^\dagger w_{jk} a_k$.

Hence, the system parametrization can be written as $G = (S, C, \Omega)$, where $C_{n\times n} = \left(c_{jk}\right)$ and $\Omega_{n\times n} = \left(\omega_{jk}\right)$. By following (100) and (103), the Heisenberg-Langevin equation for the system state vector a and input-output relations can, respectively, be simplified to

$$\dot{a}(t) = Aa(t) - C^{\dagger} S b_{in}(t) \tag{115}$$

$$b_{out} = S b_{in}(t) + Ca(t) \tag{116}$$

in which $A = -\frac{1}{2}C^{\dagger}C - i\Omega$. These linear equations can now be transferred to frequency space by computing either the Laplace or the Fourier transform, [248]. By Laplace transform, defined as the following for Re $s > 0$,

$$\hat{C}(s) = \int_0^{\infty} e^{-st} C(t)\, dt \tag{117}$$

in which C is any of our stochastic processes, the Langevin equations can be obtained as

$$\hat{a}(s) = -(sI_m - A)^{-1} C^{\dagger} S \hat{b}_{in}(s) + (sI_m - A)^{-1} a \tag{118}$$

$$\hat{b}_{out}(s) = S \hat{b}_{in}(s) + C\hat{a}(s) \cdot \tag{119}$$

Therefore, we can obtain

$$\hat{b}_{out}(s) = \Xi(s)\, \hat{b}_{in}(s) + \xi(s)\, a \tag{120}$$

in which $\xi(s) = C(sI_m - A)^{-1}$, and the transfer matrix function $\Xi(s)$ is

$$\Xi(s) = S - C(sI_m - A)^{-1} C^{\dagger} S \cdot \tag{121}$$

For the single mode cavity with $L = \sqrt{\gamma}a$, so $C = \sqrt{\gamma}$, $H = \omega a^{\dagger} a$ and $S = e^{i\phi}$, the corresponding transfer function is, [249],

$$\Xi_{cavity}(s) = e^{i\phi} \frac{s + i\omega - \frac{\gamma}{2}}{s + i\omega + \frac{\gamma}{2}}. \tag{122}$$

The frequency space framework simplifies the modelling of time delays, and non-conservative components such as quantum amplifiers, which can be modelled statistically by a Bogoliubov transformation, [187]. Moreover, frequency space makes it possible to consider the transfer function of two cascaded systems as the product of their transfer functions.

7. Robust Control

In real world, quantum systems are inevitably subjected to disturbances, uncertainties, incomplete knowledge and unknown errors, which all can be considered as uncertainties in the control field, and field-coupling coefficient, which affects the system behaviour. Hence, it is important to design a control field robust to disturbances, [250]. By applying a minimax analysis, the general formulation of quantum robust optimal control problem can be calculated, and the corresponding designed optimal control field is said to be robust to the disturbances. To attain robustness for quantum technology, and also evolve new intuitions into complex quantum plants, it is helpful to extend theories of classical robust control into quantum domain. For instance, the famous small gain theorem of classical control theory has been extended for the problem of stability of quantum feedback networks, and it has been shown that quantum feedback networks are stable if the loop gain is less than one, [251]. Moreover, some open-loop and feedback approaches such as transfer function approach can reduce noise and lead to a certain degree of robustness, [248]. Different types of robust control approaches, including H^{∞} control, [51], sliding mode control, [54], risk-sensitive control, [195, 58], and ensembel control, [59], have been proposed and introduced to the quantum realm, that shaped the early progress in this area.

7.1. H^{∞} Quantum Control

The H^{∞} controller synthesis problem was first formulated and solved in [51]. Later, a time-variant H^{∞} control problem has been solved for a class of linear quantum systems, in which a dynamic game method was employed. The

adopted technique acquired an uniformity between the quantum problem and a supplementary classical stochastic H^∞ problem. By means of the similar methodology, the finite horizon dynamic game approach has been exploited for the same class of quantum systems, and it has been shown that its solving procedure is similar to solving a corresponding deterministic continuous-time problem with imperfect state measurements. From this, one can obtain the solution to the finite horizon quantum H^∞ control problem based on a pair of generalized Riccati differential equations. The finite horizon H^∞ control problem has also been extended for the case of delayed measurement, [252]. For a class of linear quantum systems, coherent control H^∞ problem has been considered in [253]. In [253], a certain approach has been selected, providing a sufficient condition and allowing for a simple tractable solution. This approach aims to find a physically realizable augmented controller such that an appropriate linear subsystem will be the standard optimal controller. Consider the quantum system under control expressed by the noncommutative stochastic model, described similar to (11) as

$$\begin{aligned} dx(t) &= Ax(t)\,dt + \begin{bmatrix} B_0 & B_1 & B_2 \end{bmatrix} \times \begin{bmatrix} dv(t)^T & dw(t)^T & du(t)^T \end{bmatrix}^T \\ dz(t) &= C_1 x(t)\,dt + D_{12}du(t) \\ dy(t) &= C_2 x(t)\,dt + \begin{bmatrix} D_{20} & D_{21} & 0_{n_y \times n_u} \end{bmatrix} \times \begin{bmatrix} dv(t)^T & dw(t)^T & du(t)^T \end{bmatrix}^T \end{aligned} \tag{123}$$

in which $x(t)$ indicates a vector of plant variables and $x(0) = x_0$. The input $w(t)$ and $dv(t)$ represent the disturbance signal and any extra quantum noise, respectively. The control $u(t)$ is defined as

$$du(t) = \beta_u(t)dt + d\tilde{u}(t) \tag{124}$$

in which $\tilde{u}(t)$ and $\beta_u(t)$ are the noise part, and the adapted self adjoint part of $u(t)$, respectively. Also, $dv(t)$ indicates any extra noise in the plant. The vectors $v(t)$, $\tilde{w}(t)$, and $\tilde{u}(t)$ are quantum noises with semi definite Hermitian Ito matrices F_v, $F_{\tilde{w}}$, and $F_{\tilde{u}}$. The controllers are supposed to be of the form

$$\begin{aligned} d\xi(t) &= A_K\xi(t)\,dt + \begin{bmatrix} B_{K1} & B_K \end{bmatrix} \times \begin{bmatrix} dv_K(t)^T & dy(t)^T \end{bmatrix}^T, \xi(0) = \xi_0 \\ du(t) &= C_K\xi(t)\,dt + \begin{bmatrix} B_{K0} & 0_{n_u \times n_y} \end{bmatrix} \times \begin{bmatrix} dv_K(t)^T & dy(t)^T \end{bmatrix}^T \end{aligned} \tag{125}$$

in which $\xi(t) = [\xi_1(t), \ldots, \xi_{n_K}(t)]^T$ indicates a vector of self-adjoint controller variables. The noise $v_K(t) = [v_{K1}(t), \ldots, v_{KK_v}(t)]^T$ is a vector of noncommutative Wiener processes with canonical Hermitian Ito matrix F_{v_K}. In addition, at time $t = 0$, it is assumed that $x(0)$ commutes with $\xi(0)$. Here, the closed-loop system is attained by identifying $\beta_u(t) = C_K\xi(t)$, and making an interconnection between (120) and (122), so that the following system is obtained with $\eta(t) = \begin{bmatrix} x(t)^T & \xi(t)^T \end{bmatrix}^T$,

$$\begin{aligned} d\eta(t) &= \tilde{A}\eta(t)\,dt + \tilde{B}dw(t) + \tilde{G}d\tilde{v}(t) = \tilde{A}\eta(t)\,dt + \begin{bmatrix} \tilde{B} & \tilde{G} \end{bmatrix} \begin{bmatrix} dw(t) \\ d\tilde{v}(t) \end{bmatrix} \\ dz(t) &= \tilde{C}\eta(t)\,dt + \tilde{H}d\tilde{v}(t) = \tilde{C}\eta(t)\,dt + \begin{bmatrix} 0 & \tilde{H} \end{bmatrix} \begin{bmatrix} dw(t) \\ d\tilde{v}(t) \end{bmatrix} \end{aligned} \tag{126}$$

which is of the form of (11), where

$$\begin{aligned} \tilde{v}(t) &= \begin{bmatrix} v(t) \\ v_K(t) \end{bmatrix}; \quad \tilde{A} = \begin{bmatrix} A & B_2C_K \\ B_KC_2 & A_K \end{bmatrix} \\ \tilde{B} &= \begin{bmatrix} B_1 \\ B_KD_{21} \end{bmatrix}; \quad \tilde{G} = \begin{bmatrix} B_0 & B_2B_{K0} \\ B_KD_{20} & B_{K1} \end{bmatrix} \\ \tilde{C} &= \begin{bmatrix} C_1 & D_{12}C_K \end{bmatrix}; \quad \tilde{H} = \begin{bmatrix} 0 & D_{12}B_{K0} \end{bmatrix}. \end{aligned} \tag{127}$$

The H^∞ controller synthesis problem aims at finding a controller (122) for a given disturbance attenuation parameter $g > 0$, the system described in (123) satisfies

$$\int_0^t \left\langle \beta_z(s)^T \beta_z(s) + \varepsilon \eta(s)^T \eta(s) \right\rangle ds \leq \left(g^2 - \varepsilon\right) \int_0^t \left\langle \beta_w(s)^T \beta_w(s) \right\rangle ds + \mu_1 + \mu_2 t \tag{128}$$

$\forall t > 0$, and for some $\varepsilon, \mu_1, \mu_2 > 0$, where $\beta_z(t) = \tilde{C}\eta(t)$. Furthermore, in order to present results on quantum H^∞ control, the plant system (120) is required to satisfy extra suppositions well discussed in [51].

7.2. Quantum Sliding Mode Control

In the context of variable structure control, considered control inputs allow to switch between several control structures. Sliding mode control is an important method in variable structure control of classical systems particularly nonlinear ones. Since the evolution of most quantum systems is nonlinear, sliding mode control is assumed as a useful robust control tool for some quantum phenomena, [40, 107], especially due to its robustness against uncertainties, [40, 107, 54]. These perturbations can be modelled in the system Hamiltonian. For instance, in [54], bounded uncertainties are modelled as $H_\Delta = \sum_l \epsilon_l(t) H_l$, where $\epsilon_l(t) \in \mathbb{R}$, $\sqrt{\sum_l \epsilon_l^2(t)} \leq \bar{\varepsilon}\,(\bar{\varepsilon} \in \mathbb{R}^+)$ and $\{H_l\}$ indicates a set of time-independent Hamiltonian operators. To apply the idea of sliding mode control to quantum systems, one needs to introduce a sliding mode where the quantum system has desired dynamics. A sliding mode may be introduced by a functional of the state $|\psi\rangle$ and system Hamiltonian H, i.e.,

$$S(|\psi\rangle, H) = 0\cdot \tag{129}$$

For instance, an eigenstate $\left|\phi_j\right\rangle$ of internal Hamiltonian H_{int} can be chosen as a sliding surface, so $S(|\psi\rangle, H) = 1 - \left|\left\langle \psi \middle| \phi_j \right\rangle\right|^2$. If $|\psi_0\rangle$ is in the sliding mode, it can be proved that the quantum system will preserve its state in this surface under the action of the internal Hamiltonian only. Quantum sliding mode is a state space domain, in which the system has some desired properties brought by eigenstates or an invariant state subspace, such as the wave function controllable subspace, [41], or decoherence-free subspace, satisfying (129). If a quantum system state is driven into a sliding mode, the state will be preserved in the sliding surface under the action of some class of Hamiltonians determined by the sliding mode. In [54], quantum control algorithms have been presented by combining various techniques, including time-optimal control, projective measurements, and quantum amplitude amplification. Reference [56] presents a variable structure control strategy with sliding modes to control two-level quantum systems robustly. In [56], an eigenstate is identified as a sliding mode, and the Lyapunov method and projective measurements were combined to achieve the sliding mode control design. The results of [56] have been extended to take into account the imapct of uncertainties during the procedure of driving the system's state back to the sliding mode domain from outside, [254]. A sampled-data scheme to the robust control of a single qubit has been presented in [57], where the robustness is defined through a sliding mode domain and the control law is designed in an offline mode, and, then used online with a single qubit having bounded uncertainties. In this work, the system Hamiltonian and the coupling strength of the system-environment interaction are considered as two classes of uncertainties.

7.3. Quantum Risk-Sensitive Control

Risk-sensitive controllers are well-known for their robustness features against uncertainties in the model and external disturbances, [195, 255, 256]. An adaptation in the LQG cost criterion, in which the cost criterion is an average of an integral and is well known as a risk-neutral criterion, leads to the form of a linear exponential quadratic gaussian (LEQG) function resulting in the close connections between robust control and risk-sensitive control. In this average of exponential of integral problem, the cost is

$$J_{LEQG} = E\left[\exp\left(\sum_{k=0}^{M-1}\left(x_k^T P x_k + u_k^T P u_k\right) + \exp x_M^T P_M x_M\right)\right] \tag{130}$$

where P, Q, and P_M are weighting matrices. $M > 0$ is a positive integer showing a finite time interval $k = 0, \ldots, M$. A quantum generalization of (130) can be considered as

$$J^{\mu}_{\omega,0}(K) = E_{\omega,0}\left[\prod_{k=0}^{M-1}\left\langle \omega_k, e^{\mu L(u_k)}\right\rangle \left\langle \omega_M, e^{\mu N}\right\rangle\right]. \tag{131}$$

Here, the state of the quantum system is described by ω indicating a positive self-adjoint operator, $\mu > 0$ is a positive risk parameter, N is a non-negative observable, and $L(u)$ is a cost function defined to be a non-negative observable depending on the control u, which encodes the designer's control objective. N is a non-negative observable defining a cost for the final state. ω_k indicates the conditional state such that if the quantum system is in state ω_k at time k, and the applied control at this time is u_k, the measurement result y_{k+1} will be recorded, and the quantum system will be transferred to a new state ω_{k+1}. The probability of y_{k+1} is $p(y_{k+1}|u_k,\ \omega_k)$, in which $p(y|u,\ \omega) = \langle \Gamma(u,y)\,\omega, I\rangle$, and the operator $\Gamma(u,y)$ is supposed to be normalized. Hence, ω_k is indicated through

$$\omega_{k+1} = \Lambda_{\Gamma}(u_k, y_{k+1})\,\omega_k \tag{132}$$

where $\Lambda_{\Gamma}(u,y)\,\omega = \dfrac{\Gamma(u,y)\,\omega}{p(y|u,\ \omega)}$. In order to realize an optimal controller for the risk-sensitive cost (131), it is necessary to introduce a new state $\hat{\omega}$, which is in general unnormalized. To do so, one needs to utilize nonlinear operators (observables) B, and (super) operators R, which can lead to the formulation and solution of a general class of of multiplicative cost optimal control problems for quantum systems. Here, the value of B at $\hat{\omega}$ is shown by $\langle \hat{\omega}, B\rangle$, extending the notation $\langle \omega, B\rangle = tr[B\omega]$. Risk-sensitive and multiplicative cost functionals can be introduced based on (super) operator valued costs $R(u)$ satisfying the real multiplicative homogeneity condition

$$R(u)\,r\hat{\omega} = rR(u)\,\hat{\omega} \tag{133}$$

for all real numbers r, and all $\hat{\omega}$, u. An operator valued cost $R(u)$ corresponding to (131) can be expressed as

$$R(u)\,\hat{\omega} = \frac{\left\langle \hat{\omega}, e^{\mu L(u)}\right\rangle}{\langle \hat{\omega}, 1\rangle}\hat{\omega}\cdot \tag{134}$$

Although $R(u)$ is nonlinear, it satisfies the real multiplicative homogeneity property (133). To express the robustness features of the risk-sensitive controller, consider the general convex duality formula, [257],

$$\log E_P\left[e^f\right] = \sup_Q \{E_Q[f] - RE(Q\|P)\} \tag{135}$$

in which P and Q indicate probability distributions, and Q is absolutely continuous with respect to P. The relative entropy is expressed as, [156],

$$RE(Q\|P) = E_Q\left[\log\frac{dQ}{dQ}\right]. \tag{136}$$

By considering $\mu = 1/\gamma^2$, and by applying (135), the following inequality will be obtained, being $P = P_{nom}$ and $Q = P_{true}$,

$$
\begin{aligned}
&\gamma^2 \log E_{P_{true}}\left[\prod_{k=0}^{M-1}\left\langle \omega_k, e^{\mu L(u_k)}\right\rangle \left\langle \omega_M, e^{\mu N}\right\rangle\right] \\
&\geq E_{P_{true}}\left[\gamma^2 \log\left(\prod_{k=0}^{M-1}\left\langle \omega_k, e^{\mu L(u_k)}\right\rangle \left\langle \omega_M, e^{\mu N}\right\rangle\right)\right] - \gamma^2 RE\left(P_{true}\,\|P_{nom}\right) \\
&= E_{P_{true}}\left[\gamma^2\left(\sum_{k=0}^{M-1}\log\left\langle \omega_k, e^{\mu L(u_k)}\right\rangle + \log\left\langle \omega_M, e^{\mu N}\right\rangle\right)\right] - \gamma^2 RE\left(P_{true}\,\|P_{nom}\right) \\
&\geq E_{P_{true}}\left[\left(\sum_{k=0}^{M-1}\langle \omega_k, L(u_k)\rangle + \langle \omega_M, N\rangle\right)\right] - \gamma^2 RE\left(P_{true}\,\|P_{nom}\right)\cdot
\end{aligned}
\tag{137}
$$

Here, Γ_{nom} is considered to design $\hat{K}^*_{nom}$. Together, Γ_{nom}, and $\hat{K}^*_{nom}$ define the probability distribution P_{nom}. Γ_{nom} does not need to be equal to the operator for the true system Γ_{true}, in reality. The controller $\hat{K}^*_{nom}$ is applied to the true system, resulting in a probability distribution P_{true}, which is assumed to be absolutely continuous with respect to P_{nom}. In (137), the following important bound is derived

$$
J^{rn}_{P_{true}}\left(\bar{K}^*_{nom}\right) \leq \gamma^2 \log J^{rs,\gamma^2}_{P_{norm}}\left(\hat{K}^*_{nom}\right) + \gamma^2 RE\left(P_{true}\,\|P_{nom}\right) \tag{138}
$$

in which $J^{rn}_{P_{true}}\left(\bar{K}^*_{nom}\right)$ indicates the risk neutral cost criterion, $J_{\omega,0}(K) = E^K_{\omega,0}\left(\sum_{i=0}^{M-1}\langle \omega_i, L(u_i)\rangle + \langle \omega_M, N\rangle\right)$ with ω_i as the solution of the system dynamics (132). The risk neutral cost is evaluated via the true system model P_{true}, and the controller $\left(\bar{K}^*_{nom}\right)$ is designed through the nominal model P_{nom}. Equation (138) bounds the cost by two terms, the first term is related to the optimal risk-sensitive cost (131), and the second is a measure of the distance between the true and nominal systems. Robustness gain parameter $\gamma^2 = 1/\mu > 0$ is desired to be as small as possible to achieve maximum robustness, [195]. Overall, risk-sensitive controllers enjoy good performance and enhanced robustness properties, which have been characterized and proved precisely. Optimal control of quantum systems by feedback using risk-sensitive performance criteria has been well studied, [195, 255]. For instance, a quantum risk-sensitive control problem has been systematically described, where the optimal controller is a dynamical system driven by measurements giving the evolution of the risk-sensitive information state, and the optimal control actions are determined via this state and dynamic programming, [255]. Moreover, in [255], a quantum risk-sensitive control has been applied to the optimal feedback control of a two-level atom using a laser and to the problem of cooling, and confinement of a quantum mechanical oscillator undergoing position measurement. The method to formulate and solve the risk-sensitive optimal control problem has been illustrated in [256], where a risk-sensitive optimal control problem for open quantum systems has been formulated. The model used in [256] for continuously monitored open quantum systems is given by a quantum Langevin equation, and the problem is solved by means of quantum stochastic calculus and dynamic programming. In addition, the problem of quantum risk-sensitive estimation has been studied, and robustness properties of risk-sensitive filtering have been investigated, [58, 258]. Risk-sensitive filtering is known as a very efficient robust estimation method for linear and nonlinear classical systems, [34, 231]. The quantum risk-sensitive estimation problem has been studied in [58], and robustness properties of risk-sensitive filter has been analysed based on a discrete approximation model. Guaranteed-cost filtering is another robust estimation method in the classical theory whose quantum version has been presented in [258]. It has been shown that quantum guaranteed-cost filtering has superior robustness properties in comparison with optimal Kalman filter, and risk-sensitive observer, [258].

7.4. Quantum Ensemble Control

Ensemble control concerns the design of the similar control input to drive an ensemble of systems, [259]. Since the difference in parameters are demonstrated as system uncertainties, ensemble control forms a systematic branch of robust control. Many applications in quantum control involve controlling a large ensemble by means of the same control input, e.g., control of inhomogeneous quantum ensembles, [260]. A canonical problem is to develop control signals being able to steer an initial state to a target state while the ensemble depends on despersion parameters. These

control designs can compensate for the dispersion in the dynamics of the system, so they are called compensating pulse sequences, [59]. Designing excitations capable of compensating dispersion in the dynamics is well studied in NMR spectroscopy with extensions to the subject of design of robust open-loop excitations called composite pulses, [261]. These pulses have been exploited in quantum information processing to correct for systematic errors in single and two-qubit operations. For instance, the form of the composite pulse sequence has been derived in order to implement robust single-qubit gates in a three-level system, [262]. Controllability is an essential issue in quantum ensembles control, determining if the control function may exist or not, [59]. The notion of ensemble controllability has been studied in quantum mechanics for quite sometime, and its mathematical aspects have been addressed, [59, 260]. Chirped pulses for two-level systems are known to have good robustness features with regard to parameter uncertainties. Also, the effect of chirped pulses have been analysed for a class of many-level systems via adiabatic techniques, [259]. In addition, the so-called counter intuitive pulses for STIRAP (Stimulated Raman Adiabatic Passage) procedures in three-level quantum systems are known to realize a population transfer even if there exist certain unknown parameters, [263]. Let consider the controlled Schrödinger equation (1) in $\mathbb{C}^N$, given by

$$i\frac{\partial}{\partial t}\left|\psi\left(t\right)\right\rangle = H^{\alpha}\left(u\left(t\right)\right)\left|\psi\left(t\right)\right\rangle, \quad \hbar = 1 \tag{139}$$

where α is a time dependent variable belonging to a compact interval $[\alpha_0, \alpha_1] \subset \mathbb{R}$, and $u : \mathbb{R} \to U$, being U an open connected subset of $\mathbb{R}^d$, is a L^{∞} function. Each matrix $H^{\alpha}(u)$ belongs to the set of $N \times N$ Hermitian matrices, and its spectrum is denoted by $\left(\lambda_j^{\alpha}(u)\right)_{j=1}^{N}$ where $j \to \left(\lambda_j^{\alpha}(u)\right)$ is the non decreasing sequence of eigenvalues of $H^{\alpha}(u)$ repeated according to their multiplicities. The orthonormal basis of associated eigenvectors is denoted as $\left(\phi_1^{\alpha}(u), \ldots, \phi_N^{\alpha}(u)\right)$. The system (139) is ensemble approximately controllable between eigenstates if for all $\varepsilon > 0$, $j, k \in \{1, \ldots, n\}$ $n \in N$, and $u_0, u_1 \in U$ such that $\left(\lambda_j^{\alpha}(u_0)\right)$ and $\left(\lambda_k^{\alpha}(u_1)\right)$ are simple for every $\alpha \in [\alpha_0, \alpha_1]$, there exists a control $u(\cdot) : [0, T] \to U$ such that for every $\alpha \in [\alpha_0, \alpha_1]$, the solution of (139) with initial condition $\psi^{\alpha}(0) = \phi_j^{\alpha}(u_0)$ satisfies the condition, [263],

$$\left\|\psi^{\alpha}(T) - e^{i\theta}\phi_k^{\alpha}(u_1)\right\| < \varepsilon, \quad \theta \in \mathbb{R}\cdot \tag{140}$$

A quantum ensemble can consist of a large number of single quantum systems with parameter variations, [59, 264]. In addition to the tools such as sampling-based learning control, [265], or multiple-samples and mixed-strategy differential evolution, [266], which have already been introduced to quantum ensemble control, it is possible to adapt other methods since ensemble is naturally a notion in quantum statistics, [115].

7.5. Sequential Convex Programming

In quantum information processing, the robust optimization problem may be formulated as searching for design variables $\theta \in \Theta$ maximizing a measure of quantum gate performance, mainly the fidelity $\mathcal{F} \in [0, 1]$, over uncertainty parameters $\delta \in \Delta$. The set Δ can contain deterministic and random variables that may impact any component of the Hamiltonian system. Maximizing the average fidelity is one of common methods for robust control of quantum gates, [86, 60]. Consider $\mathbb{E}_{\delta\in\Delta}\{\cdot\}$ as the expected value with respect to the distribution in Δ, i.e.,

$$\mathcal{F}_{avg}(\theta) = \mathbb{E}_{\delta\in\Delta}\left\{\mathcal{F}(\theta, \delta)\right\}\cdot \tag{141}$$

Despite the application of average fidelity in several cases, such as weak random noise uncertainties, worst-case fidelity is more appropriate to estimate quantum gate errors. The worst case robust optimization (or minimax optimization) has been used in several classical problems, including nonlinear programming, [267], problems with constraints, [268], unconstrained simulation-based problems, [269], amongst others. Some of these approaches can be extended to the problem of robust control of quantum gates. The worst-case robust optimization problem for quantum gate fidelity with the aim of finding design variables that maximize the worst case fidelity is expressed as

$$\begin{aligned} &Maximize \quad \mathcal{F}_{wc}(\theta) = \min_{\delta} \mathcal{F}(\theta, \delta) \\ &Subjected \quad to \quad \theta \in \Theta, \quad \delta \in \Delta\cdot \end{aligned} \tag{142}$$

The control constraints set Θ is either convex or it can be well approximated by a convex set. However, the fidelity is not a convex function of θ for any sample $\delta \in \Delta$. Powerful methods have been developed for worst-case robust optimization with non convex cost function problems, [268]. A natural approach for solving non-convex optimization problems is sequential convex programming (SCP), [60]. Following the methods in [270], and [271], and also the works done for robust optimization in [271, 272], reference [60] has represented a SCP algorithm to identify robust control fields. It has been shown that it is possible to generate high-fidelity quantum gates with a substantial robustness against uncertainties, while simultaneously using limited control resources such as field amplitude, bandwidth, and fluence, [60]. The algorithm is initialized with the following three items:

(i) A control $\theta \in \Theta \subseteq \mathbb{R}^N$, where Θ is assumed as convex.

(ii) Noise samples $\{\delta_i \in \Delta, i = 1, \ldots, L\}$, where Δ is not necessarily convex.

(iii) A convex trust region set $\tilde{\Theta}_{trust} \subseteq \mathbb{R}^N$.

After initialization, the following steps are repeated until stopping criteria is satisfied.

- Calculate fidelities and gradients
  $\mathcal{F}(\theta, \delta_i), \quad \nabla_\theta \mathcal{F}(\theta, \delta_i) \in \mathbb{R}^N, \quad i = 1, \ldots, L.$

- Via the linearised fidelity $\mathcal{F}(\theta, \delta_i) + \tilde{\theta}^T \nabla_\theta \mathcal{F}(\theta, \delta_i)$, obtain the increment $\tilde{\theta} \in \tilde{\Theta}_{trust}$ by solving convex optimization.

- Maximize $min[\mathcal{F}(\theta, \delta_i) + \tilde{\theta}^T \nabla_\theta \mathcal{F}(\theta, \delta_i)]$
  subject to $\theta + \tilde{\theta} \in \Theta, \tilde{\theta} \in \tilde{\Theta}_{trust}$.

- Update
  if $\min_i \mathcal{F}\left(\theta + \tilde{\theta}, \delta_i\right) > \min_i \mathcal{F}(\theta, \delta_i)$, then
  substitute θ by $\left(\theta + \tilde{\theta}\right)$ and increase $\tilde{\Theta}_{trust}$
  else
  decrease $\tilde{\Theta}_{trust}$
  end if

Since SCP is a local optimization strategy, it is not possible to see whether a globally optimal solution has been obtained. In quantum computing, any improvement in fidelity is important since it can significantly reduce the physical resources needed for fault-tolerant operation. The fidelities that are usually obtained are extremely close to 1. However, in case that the maximum fidelity of 1 is achieved, it is obviously an optimal solution. SCP can be especially beneficial for finding possible improvements in the robust gate performance for different values and ranges available for both control and system designs, [60].

7.6. Quantum Dynamical Decoupling

A substantial obstacle for high-accuracy quantum control is decoherence. The lifetime of quantum information is typically not long enough to allow practical applications while it is essential to preserve quantum behaviour at the presence of noise in order to store and process quantum information. Dynamical decoupling (DD), [61], is a promising methodology to decrease decoherence by weakening the system-environment interaction with a periodic sequence of π pulses applied to the qubits, [273]. Generally, DD sequences can be divided to the following two groups introduced as dynamical decoupling sequences with a single rotation axis, Carr–Purcell–Meiboom–Gill (CPMG), [274], and Uhrig dynamical decoupling (UDD), [275], or multiple rotation axes, such as concatenated dynamical decoupling (CDD), [276], and Knill dynamical decoupling (KDD) sequences, [277].

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DD is an important scheme in quantum information processing, [278, 279, 277, 280, 281], which keeps a given input quantum state or preserve a quantum state coherent while it is combined by logical operations. For instance, reference [279] demonstrated the robustness of qubit memory error suppression through DD techniques across a variety of qubit technologies. Reference [277] has presented methods to make DD sequences robust against experimental errors while preserving good decoupling efficiency in a fluctuating environment. It has been shown that DD technique is appropriate for implementation in relatively noisy, and small-scale cloud-based quantum computers, [280]. Recently, by combining nonadiabatic holonomic quantum computation and DD, a protocol has been designed which, not only possesses the intrinsic robustness against control errors, but also protects quantum gates against environment-induced decoherence, [281]. DD consists in applying a periodic series of inversion pulses to the qubits in cycles of period τ_c to refocus the system-environment evolution. Over that period, the time-averaged system environment interaction Hamiltonian vanishes, [282]. The time average over τ_c can be calculated using average Hamiltonian theory. Other relevant parameters are the delay τ_d between pulses and the correlation time τ_e of the environment, [282]. If $\tau_d < \tau_e$, the state conservation of a single qubit is perfectly possible [283]. Although the usefulness of a decoupling strategy is dependent to the repetition rate of pulses, systems with quick fluctuating environment put huge demands on the hardware. They may be experienced in several quantum information processing implementations, [284], and demonstrate the most challenging regime to avoid decoherence, [277]. Because the accuracy of any real operation is finite, the performance of control fields used for decoupling is also limited by pulse errors, [282, 284], which can accumulate depending on the DD sequences and result in reduction of the fidelity. Therefore, one needs to analyse the corresponding errors and proper strategies to combine rotations such that the errors cancel instead of accumulate. Finite duration of real control pulses is a non ideal error implying a minimum achievable cycle time. It is stated that high order CDD, [276], or UDD, [275], sequences lose their advantages when the delays between pulses or the pulse lengths themselves are strongly constrained, [285]. Although the limitation on the cycle time decreases the maximal achievable DD performance, pulse errors can be even more destructive. The main reason of these errors is the error between the actual and the ideal amplitude of the control field which results in the deviation of rotation angle from π, normally by a few percent. Pulse imperfections also may make some DD sequences asymmetric with respect to the initial conditions. This has been shown for different DD sequences, [277, 282, 286], including CPMG, [274], which consists of identical π-pulses, and XY-4 sequence, [287], which consists of π-pulses applied along the x, and y axes. Reference [286] has reached the fidelity of 0.988 for CPMG cycles and 0.989 for the XY-4 cycles between the experimental matrices, and the ideal matrix. As seen, despite the successful implementation of DD strategies and the resulting increase of the coherence times by several orders of magnitude, [277, 282, 284], reduction of the decay rates is mainly limited by pulse imperfections, so it is necessary to make decoupling sequences robust to such defects. In quantum information processing, several possibilities have been indicated in order to generate high fidelity sequences, [277, 276]. One technique to compensate the errors is through composite pulses, [261]. Alternatively, one can design the sequence in away that the error presented by one pulse is compensated by subsequent pulses, without compensating the error of each pulse individually. The former approach is termed as using robust pulses and the latter approach as using robust or self-correcting sequences, [277]. Experiments have tested different class-A composite π-pulses, [277, 284], and it was found out that the following Knill pulse,[277],

$$(\pi)_{\pi/6+\phi} - (\pi)_{\phi} - (\pi)_{\pi/2+\phi} - (\pi)_{\phi} - (\pi)_{\pi/6+\phi} \tag{143}$$

is the most robust against flip-angle and off-resonance errors. Reference [277] has reported a comparison between standard sequences (not using robust pulses) against sequences with robust pulses. Another important development in quantum computation is the combination of DD sequences with the methods used for robust quantum gates implementation, [62]. Since DD does not need auxiliary qubits or measurements, it can be considered as an economical substitution for complementing quantum error correcting codes, [278], and reaching the error threshold for reliable quantum computation. The theoretical works in [278, 62, 288] have proposed methods to combine dynamical decoupling with quantum error correction. DD pulse sequences, [286], have also been applied to probe the noise spectrum directly, [289, 290], and detect weak magnetic fields, [291].

8. Conclusion

Quantum control mainly addresses the design of efficient manipulation methods to create desired behaviors for the systems obeying quantum-mechanical laws. In this survey, we have reviewed some of the main results and recent developments in quantum control theory, including the quantum control modelling and controllability, open- and closed-loop control strategies, and feedback control methods. Open-loop control can almost easily be derived with successful applications in many instances. Closed-loop quantum control methods may also be applied when open-loop control methods fail to act effectively. The efficiency of closed-loop control methods has been validated in several experiments. Feedback control is a widely-used strategy applied for classical control systems. However, the application of feedback control methods for quantum control problems has inevitable issues mainly caused by measurements and time delays. Finally, taking into account the importance of robustness of quantum systems for dealing with uncertainties, several methods for quantum robust control are also surveyed. Overall, based on numerous techniques developed in classical control, quantum control has a great potential to expand technology and capabilities beyond our imagination. Currently, some quantum systems can already be controlled as assessed in this paper, while still much works need to be done. More studies in this field will provide improved control that may be applicable for more quantum systems. The progress in this field needs the collaboration of experts in different fields of science, including control engineering, quantum physics, artificial intelligence, computer engineering, physical chemistry, atomic physics, to name a few.

9. Declarations

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9.2. Competing Interests

The authors declare there is no conflict of interest.

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References

[1] J. Madronero, A. Ponomarev, A. R. Carvalho, S. Wimberger, C. Viviescas, A. Kolovsky, K. Hornberger, P. Schlagheck, A. Krug, and A. Buchleitner, "Quantum chaos, transport, and control—in quantum optics," *Advances In Atomic, Molecular, and Optical Physics*, vol. 53, pp. 33–73, 2006.

[2] J. Miguel-Sánchez, A. Reinhard, E. Togan, T. Volz, A. Imamoglu, B. Besga, J. Reichel, and J. Estève, "Cavity quantum electrodynamics with charge-controlled quantum dots coupled to a fiber fabry–perot cavity," *New Journal of Physics*, vol. 15, no. 4, p. 045002, 2013.

[3] E. Bauch, C. A. Hart, J. M. Schloss, M. J. Turner, J. F. Barry, P. Kehayias, S. Singh, and R. L. Walsworth, "Ultralong dephasing times in solid-state spin ensembles via quantum control," *Physical Review X*, vol. 8, no. 3, p. 031025, 2018.

[4] J. Alonso, F. M. Leupold, B. C. Keitch, and J. Home, "Quantum control of the motional states of trapped ions through fast switching of trapping potentials," *New Journal of Physics*, vol. 15, no. 2, p. 023001, 2013.

[5] S. Amri, R. Corgier, D. Sugny, E. M. Rasel, N. Gaaloul, and E. Charron, "Optimal control of the transport of bose-einstein condensates with atom chips," *Scientific reports*, vol. 9, no. 1, pp. 1–11, 2019.

[6] G.-Q. Liu, X. Feng, N. Wang, Q. Li, and R.-B. Liu, "Coherent quantum control of nitrogen-vacancy center spins near 1000 kelvin," *Nature communications*, vol. 10, no. 1, pp. 1–8, 2019.

[7] S. Gamble, "Quantum computing: What it is, why we want it, and how we're trying to get it," in *Frontiers of Engineering: Reports on Leading-Edge Engineering from the 2018 Symposium*. National Academies Press, 2019.
[8] L. Zyga, "Physicists find quantum coherence and quantum entanglement are two sides of the same coin," *Phys.org*, 2015.
[9] V. Ramakrishna and H. Rabitz, "Relation between quantum computing and quantum controllability," *Physical Review A*, vol. 54, no. 2, p. 1715, 1996.
[10] F. Albertini and D. D'Alessandro, "Notions of controllability for quantum mechanical systems," in *Proceedings of the 40th IEEE Conference on Decision and Control (Cat. No. 01CH37228)*, vol. 2. IEEE, 2001, pp. 1589–1594.
[11] H. Rabitz, R. de Vivie-Riedle, M. Motzkus, and K. Kompa, "Whither the future of controlling quantum phenomena?" *Science*, vol. 288, no. 5467, pp. 824–828, 2000.
[12] W. S. Warren, H. Rabitz, and M. Dahleh, "Coherent control of quantum dynamics: the dream is alive," *Science*, vol. 259, no. 5101, pp. 1581–1589, 1993.
[13] R. J. Gordon, L. Zhu, and T. Seideman, "Coherent control of chemical reactions," *Accounts of Chemical Research*, vol. 32, no. 12, pp. 1007–1016, 1999.
[14] N. B. Dehaghani and F. L. Pereira, "Optimal control of quantum systems by pontryagin maximum principle," *U. Porto Journal of Engineering*, vol. 8, no. 2, pp. 194–201, 2022.
[15] ——, "High fidelity quantum state transfer by pontryagin maximum principle," *arXiv preprint arXiv:2203.04361*, 2022.
[16] ——, "A quantum optimal control problem with state constrained preserving coherence," *arXiv preprint arXiv:2203.15727*, 2022.
[17] K. W. M. Tibbetts, C. Brif, M. D. Grace, A. Donovan, D. L. Hocker, T.-S. Ho, R.-B. Wu, and H. Rabitz, "Exploring the tradeoff between fidelity and time optimal control of quantum unitary transformations," *Physical Review A*, vol. 86, no. 6, p. 062309, 2012.
[18] J. P. Palao and R. Kosloff, "Optimal control theory for unitary transformations," *Physical Review A*, vol. 68, no. 6, p. 062308, 2003.
[19] S. Cong, *Control of quantum systems: theory and methods*. John Wiley & Sons, 2014.
[20] T. Untidt, B. SCHULTE-HERBRÜGGEN, B. Luy, S. Glaser, C. Griesinger, O. Sørensen, and N. Nielsen, "Design of nmr pulse experiments with optimum sensitivity: coherence-order-selective transfer in i2s and i3s spin systems," *Molecular Physics*, vol. 95, no. 5, pp. 787–796, 1998.
[21] J.-W. Wu, C.-W. Li, T.-J. Tarn, and J. Zhang, "Optimal bang-bang control for su (1, 1) coherent states," *Physical Review A*, vol. 76, no. 5, p. 053403, 2007.
[22] U. Boscain and B. Piccoli, *Optimal syntheses for control systems on 2-D manifolds*. Springer Science & Business Media, 2003, vol. 43.
[23] S. Mancini and H. M. Wiseman, "Optimal control of entanglement via quantum feedback," *Physical Review A*, vol. 75, no. 1, p. 012330, 2007.
[24] J. Geremia and H. Rabitz, "Optimal identification of hamiltonian information by closed-loop laser control of quantum systems," *Physical review letters*, vol. 89, no. 26, p. 263902, 2002.
[25] B.-S. Chen, W.-H. Chen, F. Hsu, and W. Zhang, "Optimal tracking control design of quantum systems via tensor formal power series method," *The Open Automation and Control Systems Journal*, vol. 1, no. 1, 2008.
[26] F. Ticozzi and L. Viola, "Analysis and synthesis of attractive quantum markovian dynamics," *Automatica*, vol. 45, no. 9, pp. 2002–2009, 2009.
[27] D. Dong, C. Zhang, H. Rabitz, A. Pechen, and T.-J. Tarn, "Incoherent control of locally controllable quantum systems," *The Journal of Chemical Physics*, vol. 129, no. 15, p. 154103, 2008.
[28] L. Roa, A. Delgado, M. L. de Guevara, and A. Klimov, "Measurement-driven quantum evolution," *Physical Review A*, vol. 73, no. 1, p. 012322, 2006.
[29] K. Zhdanov, "An improvement of d-morph method for finding quantum optimal control," *arXiv preprint arXiv:1709.05364*, 2017.
[30] M. Tsubouchi and T. Momose, "Rovibrational wave-packet manipulation using shaped midinfrared femtosecond pulses toward quantum computation: Optimization of pulse shape by a genetic algorithm," *Physical Review A*, vol. 77, no. 5, p. 052326, 2008.
[31] R. Storn and K. Price, "Differential evolution–a simple and efficient heuristic for global optimization over continuous spaces," *Journal of global optimization*, vol. 11, no. 4, pp. 341–359, 1997.
[32] S. Yang, M. Wang *et al.*, "A quantum particle swarm optimization," in *Proceedings of the 2004 congress on evolutionary computation (IEEE Cat. No. 04TH8753)*, vol. 1. IEEE, 2004, pp. 320–324.
[33] R. S. Sutton and A. G. Barto, *Reinforcement learning: An introduction*. MIT press, 2018.
[34] D. Dong, C. Chen, H. Li, and T.-J. Tarn, "Quantum reinforcement learning," *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)*, vol. 38, no. 5, pp. 1207–1220, 2008.
[35] R. S. Judson and H. Rabitz, "Teaching lasers to control molecules," *Physical review letters*, vol. 68, no. 10, p. 1500, 1992.
[36] C. Brif, R. Chakrabarti, and H. Rabitz, "Control of quantum phenomena: past, present and future," *New Journal of Physics*, vol. 12, no. 7, p. 075008, 2010.
[37] Y. Sun, C. Wu, Z. Zhu, and C. Chen, "Comparison of learning methods for landscape control of open quantum systems," in *Proceeding of the 11th World Congress on Intelligent Control and Automation*. IEEE, 2014, pp. 1241–1246.
[38] X. Wang and S. G. Schirmer, "Analysis of lyapunov method for control of quantum states," *IEEE Transactions on Automatic control*, vol. 55, no. 10, pp. 2259–2270, 2010.
[39] S. Cong and F. Meng, "A survey of quantum lyapunov control methods," *The Scientific World Journal*, vol. 2013, 2013.
[40] W. Cui, Y. Pan, and D. Dong, "Variable structure control of quantum systems," in *2015 54th IEEE Conference on Decision and Control (CDC)*. IEEE, 2015, pp. 6441–6446.
[41] D. Dong and I. R. Petersen, "Variable structure control of uncontrollable quantum systems," *IFAC Proceedings Volumes*, vol. 42, no. 6, pp. 237–242, 2009.
[42] ——, "Controllability of quantum systems with switching control," *International Journal of Control*, vol. 84, no. 1, pp. 37–46, 2011.
[43] L. M. Johansen, "Quantum theory of successive projective measurements," *Physical Review A*, vol. 76, no. 1, p. 012119, 2007.
[44] A. Korotkov and D. Averin, "Continuous weak measurement of quantum coherent oscillations," *Physical Review B*, vol. 64, no. 16, p. 165310, 2001.

[45] H. M. Wiseman, "Quantum theory of continuous feedback," *Physical Review A*, vol. 49, no. 3, p. 2133, 1994.
[46] R. Ruskov and A. N. Korotkov, "Quantum feedback control of a solid-state qubit," *Physical Review B*, vol. 66, no. 4, p. 041401, 2002.
[47] C. Ahn, A. C. Doherty, and A. J. Landahl, "Continuous quantum error correction via quantum feedback control," *Physical Review A*, vol. 65, no. 4, p. 042301, 2002.
[48] J. Combes and K. Jacobs, "Rapid state reduction of quantum systems using feedback control," *Physical review letters*, vol. 96, no. 1, p. 010504, 2006.
[49] C. Hill and J. Ralph, "Weak measurement and control of entanglement generation," *Physical Review A*, vol. 77, no. 1, p. 014305, 2008.
[50] A. M. Brańczyk, P. E. Mendonça, A. Gilchrist, A. C. Doherty, and S. D. Bartlett, "Quantum control of a single qubit," *Physical Review A*, vol. 75, no. 1, p. 012329, 2007.
[51] M. R. James, H. I. Nurdin, and I. R. Petersen, "H^∞ control of linear quantum stochastic systems," *IEEE Transactions on Automatic Control*, vol. 53, no. 8, pp. 1787–1803, 2008.
[52] H. I. Nurdin, M. R. James, and I. R. Petersen, "Coherent quantum lqg control," *Automatica*, vol. 45, no. 8, pp. 1837–1846, 2009.
[53] A. Shaiju, I. R. Petersen, and M. R. James, "Guaranteed cost lqg control of uncertain linear stochastic quantum systems," in *2007 American Control Conference*. IEEE, 2007, pp. 2118–2123.
[54] D. Dong and I. R. Petersen, "Sliding mode control of quantum systems," *New Journal of Physics*, vol. 11, no. 10, p. 105033, 2009.
[55] F. Chen, R. Hou, B. Jiang, and G. Tao, "Study on fast terminal sliding mode control for a helicopter via quantum information technique and nonlinear fault observer," *International Journal of Innovative Computing, Information and Control*, vol. 9, no. 8, pp. 3437–3447, 2013.
[56] D. Dong and I. R. Petersen, "Sliding mode control of two-level quantum systems," *Automatica*, vol. 48, no. 5, pp. 725–735, 2012.
[57] D. Dong, I. R. Petersen, and H. Rabitz, "Sampled-data design for robust control of a single qubit," *IEEE Transactions on Automatic Control*, vol. 58, no. 10, pp. 2654–2659, 2013.
[58] N. Yamamoto and L. Bouten, "Quantum risk-sensitive estimation and robustness," *IEEE Transactions on Automatic Control*, vol. 54, no. 1, pp. 92–107, 2009.
[59] J.-S. Li and N. Khaneja, "Ensemble control of bloch equations," *IEEE Transactions on Automatic Control*, vol. 54, no. 3, pp. 528–536, 2009.
[60] R. L. Kosut, M. D. Grace, and C. Brif, "Robust control of quantum gates via sequential convex programming," *Physical Review A*, vol. 88, no. 5, p. 052326, 2013.
[61] W. Yang, Z.-Y. Wang, and R.-B. Liu, "Preserving qubit coherence by dynamical decoupling," *Frontiers of Physics in China*, vol. 6, no. 1, pp. 2–14, 2011.
[62] J. R. West, D. A. Lidar, B. H. Fong, and M. F. Gyure, "High fidelity quantum gates via dynamical decoupling," *Physical review letters*, vol. 105, no. 23, p. 230503, 2010.
[63] P. A. M. Dirac, *The principles of quantum mechanics*. Oxford university press, 1981, no. 27.
[64] M. A. Nielsen and I. Chuang, "Quantum computation and quantum information," 2002.
[65] H.-P. Breuer, F. Petruccione *et al.*, *The theory of open quantum systems*. Oxford University Press on Demand, 2002.
[66] R. Alicki and K. Lendi, *Quantum dynamical semigroups and applications*. Springer, 2007, vol. 717.
[67] R. Van Handel, J. K. Stockton, and H. Mabuchi, "Feedback control of quantum state reduction," *IEEE Transactions on Automatic Control*, vol. 50, no. 6, pp. 768–780, 2005.
[68] J. Zhang, R.-B. Wu, Y.-x. Liu, C.-W. Li, and T.-J. Tarn, "Quantum coherent nonlinear feedback with applications to quantum optics on chip," *IEEE transactions on automatic control*, vol. 57, no. 8, pp. 1997–2008, 2012.
[69] S. Attal, "Quantum noises," in *Open Quantum Systems II*. Springer, 2006, pp. 79–147.
[70] J. W. Clark, D. G. Lucarelli, and T.-J. Tarn, "Control of quantum systems," *International Journal of Modern Physics B*, vol. 17, no. 28, pp. 5397–5411, 2003.
[71] S. G. Schirmer, H. Fu, and A. I. Solomon, "Complete controllability of quantum systems," *Physical Review A*, vol. 63, no. 6, p. 063410, 2001.
[72] F. Albertini and D. D'Alessandro, "Notions of controllability for bilinear multilevel quantum systems," *IEEE Transactions on Automatic Control*, vol. 48, no. 8, pp. 1399–1403, 2003.
[73] L. Pinna, "On the controllability of the quantum dynamics of closed and open systems," Ph.D. dissertation, Université Paris-Saclay (ComUE), 2018.
[74] C. Altafini, "Controllability of quantum mechanical systems by root space decomposition of su (n)," *Journal of Mathematical Physics*, vol. 43, no. 5, pp. 2051–2062, 2002.
[75] G. Turinici and H. Rabitz, "Wavefunction controllability for finite-dimensional bilinear quantum systems," *Journal of Physics A: Mathematical and General*, vol. 36, no. 10, p. 2565, 2003.
[76] R. Diestel, "Graph theory 3rd ed," *Graduate texts in mathematics*, vol. 173, 2005.
[77] S. Schirmer, A. Solomon, and J. Leahy, "Degrees of controllability for quantum systems and application to atomic systems," *Journal of Physics A: Mathematical and General*, vol. 35, no. 18, p. 4125, 2002.
[78] D. Dong and I. R. Petersen, "Quantum control theory and applications: a survey," *IET Control Theory & Applications*, vol. 4, no. 12, pp. 2651–2671, 2010.
[79] D. d'Alessandro, *Introduction to quantum control and dynamics*. CRC press, 2007.
[80] J. Werschnik and E. Gross, "Quantum optimal control theory," *Journal of Physics B: Atomic, Molecular and Optical Physics*, vol. 40, no. 18, p. R175, 2007.
[81] M. James, "Optimal quantum control theory," *Annual Review of Control, Robotics, and Autonomous Systems*, vol. 4, pp. 343–367, 2021.
[82] H. O. Fattorini, *Infinite dimensional optimization and control theory*. Cambridge University Press, 1999, vol. 54.
[83] W. Zhu and H. Rabitz, "A rapid monotonically convergent iteration algorithm for quantum optimal control over the expectation value of a positive definite operator," *The Journal of Chemical Physics*, vol. 109, no. 2, pp. 385–391, 1998.
[84] Y. Maday and G. Turinici, "New formulations of monotonically convergent quantum control algorithms," *The Journal of Chemical Physics*, vol. 118, no. 18, pp. 8191–8196, 2003.
[85] O. V. Morzhin and A. N. Pechen, "Krotov method for optimal control of closed quantum systems," *Russian Mathematical Surveys*, vol. 74,

no. 5, p. 851, 2019.

[86] N. Khaneja, T. Reiss, C. Kehlet, T. Schulte-Herbrüggen, and S. J. Glaser, "Optimal control of coupled spin dynamics: design of nmr pulse sequences by gradient ascent algorithms," *Journal of magnetic resonance*, vol. 172, no. 2, pp. 296–305, 2005.

[87] F. K. Wilhelm, S. Kirchhoff, S. Machnes, N. Wittler, and D. Sugny, "An introduction into optimal control for quantum technologies," *arXiv preprint arXiv:2003.10132*, 2020.

[88] J. Sørensen, M. Aranburu, T. Heinzel, and J. Sherson, "Quantum optimal control in a chopped basis: Applications in control of bose-einstein condensates," *Physical Review A*, vol. 98, no. 2, p. 022119, 2018.

[89] C. Altafini and F. Ticozzi, "Modeling and control of quantum systems: an introduction," *IEEE Transactions on Automatic Control*, vol. 57, no. 8, pp. 1898–1917, 2012.

[90] A. Sinatra, F. Castelli, L. Lugiato, P. Grangier, and J. Poizat, "Effective two-level model versus three-level model," *Quantum and Semiclassical Optics: Journal of the European Optical Society Part B*, vol. 7, no. 3, p. 405, 1995.

[91] I. Kuprov and C. T. Rodgers, "Derivatives of spin dynamics simulations," *The Journal of chemical physics*, vol. 131, no. 23, p. 234108, 2009.

[92] R. Fisher, "Optimal control of multi-level quantum systems," Ph.D. dissertation, Technische Universität München, 2010.

[93] H. Yuan and N. Khaneja, "Time optimal control of coupled qubits under nonstationary interactions," *Physical Review A*, vol. 72, no. 4, p. 040301, 2005.

[94] D. D'alessandro and M. Dahleh, "Optimal control of two-level quantum systems," *IEEE Transactions on Automatic Control*, vol. 46, no. 6, pp. 866–876, 2001.

[95] Y. Lou and S. Cong, "State transfer control of quantum systems on the bloch sphere," *Journal of Systems Science and Complexity*, vol. 24, no. 3, p. 506, 2011.

[96] U. Boscain and P. Mason, "Time minimal trajectories for two-level quantum systems with drift," in *Proceedings of the 44th IEEE conference on decision and control*. IEEE, 2005, pp. 3188–3193.

[97] ——, "Time minimal trajectories for a spin 1/ 2 particle in a magnetic field," *Journal of Mathematical Physics*, vol. 47, no. 6, p. 062101, 2006.

[98] N. Khaneja, R. Brockett, and S. J. Glaser, "Time optimal control in spin systems," *Physical Review A*, vol. 63, no. 3, p. 032308, 2001.

[99] D. Dong, J. Lam, and T. Tarn, "Rapid incoherent control of quantum systems based on continuous measurements and reference model," *IET control theory & applications*, vol. 3, no. 2, pp. 161–169, 2009.

[100] C. Lin, Y. Ma, and D. Sels, "Optimal control for quantum metrology via pontryagin's principle," *Physical Review A*, vol. 103, no. 5, p. 052607, 2021.

[101] D. D'Alessandro, "Optimal evaluation of generalized euler angles with applications to control," *Automatica*, vol. 40, no. 11, pp. 1997–2002, 2004.

[102] C. Altafini, "Coherent control of open quantum dynamical systems," *Physical Review A*, vol. 70, no. 6, p. 062321, 2004.

[103] I. Oliveira, R. Sarthour Jr, T. Bonagamba, E. Azevedo, and J. C. Freitas, *NMR quantum information processing*. Elsevier, 2011.

[104] R. Romano and D. D'Alessandro, "Incoherent control and entanglement for two-dimensional coupled systems," *Physical Review A*, vol. 73, no. 2, p. 022323, 2006.

[105] ——, "Environment-mediated control of a quantum system," *Physical review letters*, vol. 97, no. 8, p. 080402, 2006.

[106] D. Burgarth and V. Giovannetti, "Full control by locally induced relaxation," *Physical review letters*, vol. 99, no. 10, p. 100501, 2007.

[107] R. V. Mendes and V. Man'ko, "Quantum control and the strocchi map," *Physical Review A*, vol. 67, no. 5, p. 053404, 2003.

[108] S. Ashhab and F. Nori, "Control-free control: Manipulating a quantum system using only a limited set of measurements," *Physical Review A*, vol. 82, no. 6, p. 062103, 2010.

[109] K. Jacobs, "Feedback control using only quantum back-action," *New Journal of Physics*, vol. 12, no. 4, p. 043005, 2010.

[110] S. Tanaka and N. Yamamoto, "Robust adaptive measurement scheme for qubit-state preparation," *Physical Review A*, vol. 86, no. 6, p. 062331, 2012.

[111] K. Jacobs, "How to project qubits faster using quantum feedback," *Physical Review A*, vol. 67, no. 3, p. 030301, 2003.

[112] R. Raussendorf and H. J. Briegel, "A one-way quantum computer," *Physical Review Letters*, vol. 86, no. 22, p. 5188, 2001.

[113] D. K. Burgarth, P. Facchi, V. Giovannetti, H. Nakazato, S. Pascazio, and K. Yuasa, "Exponential rise of dynamical complexity in quantum computing through projections," *Nature communications*, vol. 5, no. 1, pp. 1–6, 2014.

[114] S. Hacohen-Gourgy, L. P. García-Pintos, L. S. Martin, J. Dressel, and I. Siddiqi, "Incoherent qubit control using the quantum zeno effect," *Physical review letters*, vol. 120, no. 2, p. 020505, 2018.

[115] C. Chen, L.-C. Wang, and Y. Wang, "Closed-loop and robust control of quantum systems," *The scientific world journal*, vol. 2013, 2013.

[116] D. Dong, "Learning control of quantum systems," *arXiv preprint arXiv:2101.07461*, 2021.

[117] R. Chakrabarti and H. Rabitz, "Quantum control landscapes," *International Reviews in Physical Chemistry*, vol. 26, no. 4, pp. 671–735, 2007.

[118] R.-B. Wu, B. Chu, D. H. Owens, and H. Rabitz, "Data-driven gradient algorithm for high-precision quantum control," *Physical Review A*, vol. 97, no. 4, p. 042122, 2018.

[119] C.-C. Shu, T.-S. Ho, X. Xing, and H. Rabitz, "Frequency domain quantum optimal control under multiple constraints," *Physical Review A*, vol. 93, no. 3, p. 033417, 2016.

[120] A. Rothman, T.-S. Ho, and H. Rabitz, "Observable-preserving control of quantum dynamics over a family of related systems," *Physical Review A*, vol. 72, no. 2, p. 023416, 2005.

[121] M. D. Jegen, M. E. Everett, and A. Schultz, "Using homotopy to invert geophysical data," *Geophysics*, vol. 66, no. 6, pp. 1749–1760, 2001.

[122] S. Kirkpatrick, C. D. Gelatt, and M. P. Vecchi, "Optimization by simulated annealing," *science*, vol. 220, no. 4598, pp. 671–680, 1983.

[123] M. Dorigo, M. Birattari, and T. Stutzle, "Ant colony optimization," *IEEE computational intelligence magazine*, vol. 1, no. 4, pp. 28–39, 2006.

[124] P. Moscato *et al.*, "On evolution, search, optimization, genetic algorithms and martial arts: Towards memetic algorithms," *Caltech concurrent computation program, C3P Report*, vol. 826, p. 1989, 1989.

[125] H. Ma, C. Chen, and D. Dong, "Differential evolution with equally-mixed strategies for robust control of open quantum systems," in *2015 IEEE international conference on systems, man, and cybernetics*. IEEE, 2015, pp. 2055–2060.
[126] M. Pant, H. Zaheer, L. Garcia-Hernandez, A. Abraham *et al.*, "Differential evolution: A review of more than two decades of research," *Engineering Applications of Artificial Intelligence*, vol. 90, p. 103479, 2020.
[127] J. Kennedy and R. Eberhart, "Particle swarm optimization," in *Proceedings of ICNN'95-international conference on neural networks*, vol. 4. IEEE, 1995, pp. 1942–1948.
[128] D. Zeidler, S. Frey, K.-L. Kompa, and M. Motzkus, "Evolutionary algorithms and their application to optimal control studies," *Physical Review A*, vol. 64, no. 2, p. 023420, 2001.
[129] F. Glover and G. A. Kochenberger, *Handbook of metaheuristics*. Springer Science & Business Media, 2003.
[130] E. Elbeltagi, T. Hegazy, and D. Grierson, "Comparison among five evolutionary-based optimization algorithms," *Advanced engineering informatics*, vol. 19, no. 1, pp. 43–53, 2005.
[131] E. Zahedinejad, S. Schirmer, and B. C. Sanders, "Evolutionary algorithms for hard quantum control," *Physical Review A*, vol. 90, no. 3, p. 032310, 2014.
[132] V. Kachitvichyanukul, "Comparison of three evolutionary algorithms: Ga, pso, and de," *Industrial Engineering and Management Systems*, vol. 11, no. 3, pp. 215–223, 2012.
[133] S. Yang, M. Wang, and L. Jiao, "A quantum particle swarm optimization," in *Proceedings of the 2004 Congress on Evolutionary Computation (IEEE Cat. No. 04TH8753)*, vol. 1. IEEE, 2004, pp. 320–324.
[134] J. Sun, W. Xu, and W. Fang, "Quantum-behaved particle swarm optimization algorithm with controlled diversity," in *International Conference on Computational Science*. Springer, 2006, pp. 847–854.
[135] K. V. Price, "Differential evolution," in *Handbook of optimization*. Springer, 2013, pp. 187–214.
[136] S. Das and P. N. Suganthan, "Differential evolution: A survey of the state-of-the-art," *IEEE transactions on evolutionary computation*, vol. 15, no. 1, pp. 4–31, 2010.
[137] S. Das, S. S. Mullick, and P. N. Suganthan, "Recent advances in differential evolution - an updated survey," *Swarm and Evolutionary Computation*, vol. 27, pp. 1–30, 2016.
[138] X. Yang, J. Li, and X. Peng, "An improved differential evolution algorithm for learning high-fidelity quantum controls," *Science Bulletin*, vol. 64, no. 19, pp. 1402–1408, 2019.
[139] E. Zahedinejad, J. Ghosh, and B. C. Sanders, "High-fidelity single-shot toffoli gate via quantum control," *Physical review letters*, vol. 114, no. 20, p. 200502, 2015.
[140] ——, "Designing high-fidelity single-shot three-qubit gates: a machine-learning approach," *Physical Review Applied*, vol. 6, no. 5, p. 054005, 2016.
[141] T. Kondo and K. Ito, "A reinforcement learning with evolutionary state recruitment strategy for autonomous mobile robots control," *Robotics and Autonomous Systems*, vol. 46, no. 2, pp. 111–124, 2004.
[142] C. Chen, D. Dong, and Z. Chen, "Grey reinforcement learning for incomplete information processing," in *International Conference on Theory and Applications of Models of Computation*. Springer, 2006, pp. 399–407.
[143] P. He and S. Jagannathan, "Reinforcement learning-based output feedback control of nonlinear systems with input constraints," *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)*, vol. 35, no. 1, pp. 150–154, 2005.
[144] Y. Shahar, "A framework for knowledge-based temporal abstraction," *Artificial intelligence*, vol. 90, no. 1-2, pp. 79–133, 1997.
[145] B. Atal, "Efficient coding of lpc parameters by temporal decomposition," in *ICASSP'83. IEEE International Conference on Acoustics, Speech, and Signal Processing*, vol. 8. IEEE, 1983, pp. 81–84.
[146] J. A. Flanagan, "Self-organisation in kohonen's som," *Neural networks*, vol. 9, no. 7, pp. 1185–1197, 1996.
[147] C. Szepesvári, "Algorithms for reinforcement learning," *Synthesis lectures on artificial intelligence and machine learning*, vol. 4, no. 1, pp. 1–103, 2010.
[148] A. J. Smith, "Applications of the self-organising map to reinforcement learning," *Neural networks*, vol. 15, no. 8-9, pp. 1107–1124, 2002.
[149] C. J. Watkins and P. Dayan, "Q-learning," *Machine learning*, vol. 8, no. 3-4, pp. 279–292, 1992.
[150] P. Y. Glorennec and L. Jouffe, "Fuzzy q-learning," in *Proceedings of 6th international fuzzy systems conference*, vol. 2. IEEE, 1997, pp. 659–662.
[151] S. G. Tzafestas and G. G. Rigatos, "Fuzzy reinforcement learning control for compliance tasks of robotic manipulators," *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)*, vol. 32, no. 1, pp. 107–113, 2002.
[152] M. J. Er and C. Deng, "Online tuning of fuzzy inference systems using dynamic fuzzy q-learning," *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)*, vol. 34, no. 3, pp. 1478–1489, 2004.
[153] D. Vengerov, N. Bambos, and H. R. Berenji, "A fuzzy reinforcement learning approach to power control in wireless transmitters," *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)*, vol. 35, no. 4, pp. 768–778, 2005.
[154] C. Chen, H.-X. Li, and D. Dong, "Hybrid control for robot navigation-a hierarchical q-learning algorithm," *IEEE Robotics & Automation Magazine*, vol. 15, no. 2, pp. 37–47, 2008.
[155] S. Whiteson, "Evolutionary function approximation for reinforcement learning," *Journal of Machine Learning Research*, vol. 7, 2006.
[156] M. A. Nielsen and I. L. Chuang, "Quantum computation and quantum information," *Phys. Today*, vol. 54, no. 2, p. 60, 2001.
[157] A. Ekert and R. Jozsa, "Quantum computation and shor's factoring algorithm," *Reviews of Modern Physics*, vol. 68, no. 3, p. 733, 1996.
[158] L. M. Vandersypen, M. Steffen, G. Breyta, C. S. Yannoni, M. H. Sherwood, and I. L. Chuang, "Experimental realization of shor's quantum factoring algorithm using nuclear magnetic resonance," *Nature*, vol. 414, no. 6866, pp. 883–887, 2001.
[159] L. K. Grover, "Quantum mechanics helps in searching for a needle in a haystack," *Physical review letters*, vol. 79, no. 2, p. 325, 1997.
[160] M. O. Scully and M. Zubairy, "Quantum optical implementation of grover's algorithm," *Proceedings of the National Academy of Sciences*, vol. 98, no. 17, pp. 9490–9493, 2001.
[161] J. A. Jones, "Fast searches with nuclear magnetic resonance computers," *Science*, vol. 280, no. 5361, pp. 229–229, 1998.
[162] J. Zhao and R.-H. Xie, "Genetic algorithms for the geometry optimization of atomic and molecular clusters," *Journal of Computational and Theoretical Nanoscience*, vol. 1, no. 2, pp. 117–131, 2004.

[163] G. G. Rigatos and S. G. Tzafestas, "Parallelization of a fuzzy control algorithm using quantum computation," *IEEE Transactions on Fuzzy Systems*, vol. 10, no. 4, pp. 451–460, 2002.
[164] T. Hogg and D. Portnov, "Quantum optimization," *Information Sciences*, vol. 128, no. 3-4, pp. 181–197, 2000.
[165] S. Naguleswaran and L. B. White, "Quantum search in stochastic planning," in *Noise and Information in Nanoelectronics, Sensors, and Standards III*, vol. 5846. International Society for Optics and Photonics, 2005, pp. 34–45.
[166] D. Dong, C. Chen, J. Chu, and T.-J. Tarn, "Robust quantum-inspired reinforcement learning for robot navigation," *IEEE/ASME transactions on mechatronics*, vol. 17, no. 1, pp. 86–97, 2010.
[167] M. Bukov, A. G. Day, D. Sels, P. Weinberg, A. Polkovnikov, and P. Mehta, "Reinforcement learning in different phases of quantum control," *Physical Review X*, vol. 8, no. 3, p. 031086, 2018.
[168] M. Y. Niu, S. Boixo, V. N. Smelyanskiy, and H. Neven, "Universal quantum control through deep reinforcement learning," *npj Quantum Information*, vol. 5, no. 1, pp. 1–8, 2019.
[169] T. Fösel, P. Tighineanu, T. Weiss, and F. Marquardt, "Reinforcement learning with neural networks for quantum feedback," *Physical Review X*, vol. 8, no. 3, p. 031084, 2018.
[170] C.-L. Chen and D.-Y. Dong, "Superposition-inspired reinforcement learning and quantum reinforcement learning," in *Reinforcement Learning*. IntechOpen, 2008.
[171] P. Kwiat, J. Mitchell, P. Schwindt, and A. White, "Grover's search algorithm: an optical approach," *Journal of Modern Optics*, vol. 47, no. 2-3, pp. 257–266, 2000.
[172] C. Chen, D. Dong, H.-X. Li, J. Chu, and T.-J. Tarn, "Fidelity-based probabilistic q-learning for control of quantum systems," *IEEE transactions on neural networks and learning systems*, vol. 25, no. 5, pp. 920–933, 2013.
[173] P. Vettori, "On the convergence of a feedback control strategy for multilevel quantum systems," in *Proceedings of the MTNS Conference*, 2002, pp. 1–6.
[174] K. Zyczkowski and W. Slomczynski, "The monge metric on the sphere and geometry of quantum states," *Journal of Physics A: Mathematical and General*, vol. 34, no. 34, p. 6689, 2001.
[175] J. Wen, S. Cong, and X. Zou, "Realization of quantum hadamard gate based on lyapunov method," in *Proceedings of the 10th World Congress on Intelligent Control and Automation*. IEEE, 2012, pp. 5096–5101.
[176] S. M. Nejad and M. Mehmandoost, "Realization of quantum hadamard gate by applying optimal control fields to a spin qubit," in *2010 2nd International Conference on Mechanical and Electronics Engineering*, vol. 2. IEEE, 2010, pp. V2–292.
[177] M. Sugawara, "General formulation of locally designed coherent control theory for quantum system," *The Journal of chemical physics*, vol. 118, no. 15, pp. 6784–6800, 2003.
[178] W. Liang, N. H. Amini, and P. Mason, "Feedback exponential stabilization of ghz states of multi-qubit systems," *arXiv preprint arXiv:2011.00097*, 2020.
[179] M. Mirrahimi, G. Turinici, and P. Rouchon, "Reference trajectory tracking for locally designed coherent quantum controls," *The Journal of Physical Chemistry A*, vol. 109, no. 11, pp. 2631–2637, 2005.
[180] S. Grivopoulos and B. Bamieh, "Lyapunov-based control of quantum systems," in *42nd IEEE International Conference on Decision and Control (IEEE Cat. No. 03CH37475)*, vol. 1. IEEE, 2003, pp. 434–438.
[181] M. Mirrahimi, P. Rouchon, and G. Turinici, "Lyapunov control of bilinear schrödinger equations," *Automatica*, vol. 41, no. 11, pp. 1987–1994, 2005.
[182] C. Altafini, "Feedback stabilization of isospectral control systems on complex flag manifolds: application to quantum ensembles," *IEEE Transactions on Automatic Control*, vol. 52, no. 11, pp. 2019–2028, 2007.
[183] ——, "Feedback control of spin systems," *Quantum Information Processing*, vol. 6, no. 1, pp. 9–36, 2007.
[184] S. H. Tersigni, P. Gaspard, and S. A. Rice, "On using shaped light pulses to control the selectivity of product formation in a chemical reaction: An application to a multiple level system," *The Journal of chemical physics*, vol. 93, no. 3, pp. 1670–1680, 1990.
[185] J. Cohen and M. Mirrahimi, "Dissipation-induced continuous quantum error correction for superconducting circuits," *Physical Review A*, vol. 90, no. 6, p. 062344, 2014.
[186] K. Maruyama, F. Nori, and V. Vedral, "Colloquium: The physics of maxwell's demon and information," *Reviews of Modern Physics*, vol. 81, no. 1, p. 1, 2009.
[187] J. Zhang, Y.-x. Liu, R.-B. Wu, K. Jacobs, and F. Nori, "Quantum feedback: theory, experiments, and applications," *Physics Reports*, vol. 679, pp. 1–60, 2017.
[188] K. Jacobs and D. A. Steck, "A straightforward introduction to continuous quantum measurement," *Contemporary Physics*, vol. 47, no. 5, pp. 279–303, 2006.
[189] M. B. Mensky, *Continuous quantum measurements and path integrals*. CRC Press, 1993.
[190] H.-S. Goan, G. J. Milburn, H. M. Wiseman, and H. B. Sun, "Continuous quantum measurement of two coupled quantum dots using a point contact: A quantum trajectory approach," *Physical Review B*, vol. 63, no. 12, p. 125326, 2001.
[191] J. Gambetta, A. Blais, M. Boissonneault, A. A. Houck, D. Schuster, and S. M. Girvin, "Quantum trajectory approach to circuit qed: Quantum jumps and the zeno effect," *Physical Review A*, vol. 77, no. 1, p. 012112, 2008.
[192] J. Jing and T. Yu, "Non-markovian relaxation of a three-level system: quantum trajectory approach," *Physical review letters*, vol. 105, no. 24, p. 240403, 2010.
[193] Y. Fuji and Y. Ashida, "Measurement-induced quantum criticality under continuous monitoring," *Physical Review B*, vol. 102, no. 5, p. 054302, 2020.
[194] L. Bouten, R. Van Handel, and M. R. James, "An introduction to quantum filtering," *SIAM Journal on Control and Optimization*, vol. 46, no. 6, pp. 2199–2241, 2007.
[195] M. James, "Risk-sensitive optimal control of quantum systems," *Physical Review A*, vol. 69, no. 3, p. 032108, 2004.
[196] P. Rouchon and J. F. Ralph, "Efficient quantum filtering for quantum feedback control," *Physical Review A*, vol. 91, no. 1, p. 012118, 2015.
[197] J. Dalibard, Y. Castin, and K. Mølmer, "Wave-function approach to dissipative processes in quantum optics," *Physical review letters*, vol. 68, no. 5, p. 580, 1992.

[198] W. K. Wootters, "Quantum measurements and finite geometry," *Foundations of Physics*, vol. 36, no. 1, pp. 112–126, 2006.
[199] G. Gillett, R. Dalton, B. Lanyon, M. Almeida, M. Barbieri, G. J. Pryde, J. Obrien, K. Resch, S. Bartlett, and A. White, "Experimental feedback control of quantum systems using weak measurements," *Physical review letters*, vol. 104, no. 8, p. 080503, 2010.
[200] J. Zhang, Y.-x. Liu, R.-B. Wu, C.-W. Li, and T.-J. Tarn, "Transition from weak to strong measurements by nonlinear quantum feedback control," *Physical Review A*, vol. 82, no. 2, p. 022101, 2010.
[201] G. M. D'Ariano, M. G. Paris, and M. F. Sacchi, "Quantum tomography," *Advances in Imaging and Electron Physics*, vol. 128, pp. 206–309, 2003.
[202] L. Artiles, R. Gill, and M. Guţă, "An invitation to quantum tomography," *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, vol. 67, no. 1, pp. 109–134, 2005.
[203] R. O'Donnell and J. Wright, "Efficient quantum tomography," in *Proceedings of the forty-eighth annual ACM symposium on Theory of Computing*, 2016, pp. 899–912.
[204] D. Yang, A. Grankin, L. M. Sieberer, D. V. Vasilyev, and P. Zoller, "Quantum non-demolition measurement of a many-body hamiltonian," *Nature communications*, vol. 11, no. 1, pp. 1–8, 2020.
[205] T. Nakajima, A. Noiri, J. Yoneda, M. R. Delbecq, P. Stano, T. Otsuka, K. Takeda, S. Amaha, G. Allison, K. Kawasaki *et al.*, "Quantum non-demolition measurement of an electron spin qubit," *Nature nanotechnology*, vol. 14, no. 6, pp. 555–560, 2019.
[206] M. Woolley, A. Doherty, and G. Milburn, "Continuous quantum nondemolition measurement of fock states of a nanoresonator using feedback-controlled circuit qed," *Physical Review B*, vol. 82, no. 9, p. 094511, 2010.
[207] J. Geremia, J. K. Stockton, and H. Mabuchi, "Real-time quantum feedback control of atomic spin-squeezing," *Science*, vol. 304, no. 5668, pp. 270–273, 2004.
[208] R. Inoue, R. Namiki, T. Sagawa, Y. Takahashi *et al.*, "Unconditional quantum-noise suppression via measurement-based quantum feedback," *Physical review letters*, vol. 110, no. 16, p. 163602, 2013.
[209] L. Gregory, "Quantum filtering theory and the filtering interpretation," in *AIP Conference Proceedings*, vol. 750, no. 1. American Institute of Physics, 2005, pp. 95–103.
[210] C. Gardiner, P. Zoller, and P. Zoller, *Quantum noise: a handbook of Markovian and non-Markovian quantum stochastic methods with applications to quantum optics*. Springer Science and Business Media, 2004.
[211] C. W. Gardiner and M. J. Collett, "Input and output in damped quantum systems: Quantum stochastic differential equations and the master equation," *Physical Review A*, vol. 31, no. 6, p. 3761, 1985.
[212] J. E. Gough, M. R. James, and H. I. Nurdin, "Single photon quantum filtering using non-markovian embeddings," *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, vol. 370, no. 1979, pp. 5408–5421, 2012.
[213] Z. Dong, G. Zhang, and N. H. Amini, "Quantum filtering for a two-level atom driven by two counter-propagating photons," *Quantum Information Processing*, vol. 18, no. 5, pp. 1–27, 2019.
[214] V. N. Kolokoltsov, "The law of large numbers for quantum stochastic filtering and control of many particle systems," *arXiv preprint arXiv:2008.07375*, 2020.
[215] K. P. Knutsen, J. C. Johnson, A. E. Miller, P. B. Petersen, and R. J. Saykally, "High spectral resolution multiplex cars spectroscopy using chirped pulses," *Chemical Physics Letters*, vol. 387, no. 4-6, pp. 436–441, 2004.
[216] H. M. Wiseman and G. J. Milburn, *Quantum measurement and control*. Cambridge university press, 2009.
[217] H. M. Wiseman, S. Mancini, and J. Wang, "Bayesian feedback versus markovian feedback in a two-level atom," *Physical Review A*, vol. 66, no. 1, p. 013807, 2002.
[218] J. Wang and H. M. Wiseman, "Feedback-stabilization of an arbitrary pure state of a two-level atom," *Physical Review A*, vol. 64, no. 6, p. 063810, 2001.
[219] Y. Li, B. Luo, and H. Guo, "Entanglement and quantum discord dynamics of two atoms under practical feedback control," *Physical Review A*, vol. 84, no. 1, p. 012316, 2011.
[220] A. Carvalho, A. Reid, and J. Hope, "Controlling entanglement by direct quantum feedback," *Physical Review A*, vol. 78, no. 1, p. 012334, 2008.
[221] L. Thomsen, S. Mancini, and H. M. Wiseman, "Spin squeezing via quantum feedback," *Physical Review A*, vol. 65, no. 6, p. 061801, 2002.
[222] F. Ticozzi and L. Viola, "Quantum markovian subsystems: invariance, attractivity, and control," *IEEE Transactions on Automatic Control*, vol. 53, no. 9, pp. 2048–2063, 2008.
[223] C. Ahn, H. M. Wiseman, and G. J. Milburn, "Quantum error correction for continuously detected errors," *Physical Review A*, vol. 67, no. 5, p. 052310, 2003.
[224] S. S. Ge, T. L. Vu, and T. H. Lee, "Quantum measurement-based feedback control: a nonsmooth time delay control approach," *SIAM Journal on Control and Optimization*, vol. 50, no. 2, pp. 845–863, 2012.
[225] K. Nishio, K. Kashima, and J.-i. Imura, "Effects of time delay in feedback control of linear quantum systems," *Physical Review A*, vol. 79, no. 6, p. 062105, 2009.
[226] J. K. Stockton, R. Van Handel, and H. Mabuchi, "Deterministic dicke-state preparation with continuous measurement and control," *Physical Review A*, vol. 70, no. 2, p. 022106, 2004.
[227] M. Sarovar, H.-S. Goan, T. Spiller, and G. Milburn, "High-fidelity measurement and quantum feedback control in circuit qed," *Physical Review A*, vol. 72, no. 6, p. 062327, 2005.
[228] Z. Liu, L. Kuang, K. Hu, L. Xu, S. Wei, L. Guo, and X.-Q. Li, "Deterministic creation and stabilization of entanglement in circuit qed by homodyne-mediated feedback control," *Physical Review A*, vol. 82, no. 3, p. 032335, 2010.
[229] W. Feng, P. Wang, X. Ding, L. Xu, and X.-Q. Li, "Generating and stabilizing the greenberger-horne-zeilinger state in circuit qed: Joint measurement, zeno effect, and feedback," *Physical Review A*, vol. 83, no. 4, p. 042313, 2011.
[230] A. C. Doherty and K. Jacobs, "Feedback control of quantum systems using continuous state estimation," *Physical Review A*, vol. 60, no. 4, p. 2700, 1999.
[231] A. C. Doherty, S. Habib, K. Jacobs, H. Mabuchi, and S. M. Tan, "Quantum feedback control and classical control theory," *Physical Review A*, vol. 62, no. 1, p. 012105, 2000.

[232] D. A. Steck, K. Jacobs, H. Mabuchi, T. Bhattacharya, and S. Habib, "Quantum feedback control of atomic motion in an optical cavity," *Physical review letters*, vol. 92, no. 22, p. 223004, 2004.
[233] K. Jacobs, J. Finn, and S. Vinjanampathy, "Real-time feedback control of a mesoscopic superposition," *Physical Review A*, vol. 83, no. 4, p. 041801, 2011.
[234] N. Galioto and A. A. Gorodetsky, "Bayesian identification of hamiltonian dynamics from symplectic data," in *2020 59th IEEE Conference on Decision and Control (CDC)*. IEEE, 2020, pp. 1190–1195.
[235] A. S. Holevo, *Probabilistic and statistical aspects of quantum theory*. Springer Science & Business Media, 2011, vol. 1.
[236] M. F. Emzir, M. J. Woolley, and I. R. Petersen, "A quantum extended kalman filter," *Journal of Physics A: Mathematical and Theoretical*, vol. 50, no. 22, p. 225301, 2017.
[237] S. M. Ross, *Introduction to stochastic dynamic programming*. Academic press, 2014.
[238] L. Bouten and R. V. Handel, "On the separation principle in quantum control," in *Quantum stochastics and information: statistics, filtering and control*. World Scientific, 2008, pp. 206–238.
[239] S. Lloyd, "Coherent quantum feedback," *Physical Review A*, vol. 62, no. 2, p. 022108, 2000.
[240] R. J. Nelson, Y. Weinstein, D. Cory, and S. Lloyd, "Experimental demonstration of fully coherent quantum feedback," *Physical Review Letters*, vol. 85, no. 14, p. 3045, 2000.
[241] H. M. Wiseman and G. J. Milburn, "All-optical versus electro-optical quantum-limited feedback," *Physical Review A*, vol. 49, no. 5, p. 4110, 1994.
[242] S. Lloyd, "Quantum controllers for quantum systems," *arXiv preprint quant-ph/9703042*, 1997.
[243] N. B. Dehaghani and A. H. Alizadeh, "Design, simulation and optimization of a nonreciprocal guided-wave optical isolator based on mach-zehnder interferometer," *Journal of Physics Communications*, vol. 3, no. 12, p. 125006, 2019.
[244] M. Collett and C. Gardiner, "Squeezing of intracavity and traveling-wave light fields produced in parametric amplification," *Physical Review A*, vol. 30, no. 3, p. 1386, 1984.
[245] J. Gough and M. R. James, "The series product and its application to quantum feedforward and feedback networks," *IEEE Transactions on Automatic Control*, vol. 54, no. 11, pp. 2530–2544, 2009.
[246] R. L. Hudson and K. R. Parthasarathy, "Quantum ito's formula and stochastic evolutions," *Communications in mathematical physics*, vol. 93, no. 3, pp. 301–323, 1984.
[247] J. Gough and M. R. James, "Quantum feedback networks: Hamiltonian formulation," *Communications in Mathematical Physics*, vol. 287, no. 3, pp. 1109–1132, 2009.
[248] M. Yanagisawa and H. Kimura, "Transfer function approach to quantum control-part ii: Control concepts and applications," *IEEE Transactions on Automatic Control*, vol. 48, no. 12, pp. 2121–2132, 2003.
[249] J. E. Gough, M. R. James, and H. I. Nurdin, "Squeezing components in linear quantum feedback networks," *Physical Review A*, vol. 81, no. 2, p. 023804, 2010.
[250] H. Zhang and H. Rabitz, "Robust optimal control of quantum molecular systems in the presence of disturbances and uncertainties," *Physical Review A*, vol. 49, no. 4, p. 2241, 1994.
[251] C. D'Helon and M. James, "Stability, gain, and robustness in quantum feedback networks," *Physical Review A*, vol. 73, no. 5, p. 053803, 2006.
[252] A. I. Maalouf and I. R. Petersen, "Finite horizon h infinity control for a class of linear quantum measurement delayed systems: A dynamic game approach," *SIAM Journal on Control and Optimization*, vol. 52, no. 3, pp. 1787–1808, 2014.
[253] ——, "Coherent H^∞ control for a class of annihilation operator linear quantum systems," *IEEE Transactions on Automatic Control*, vol. 56, no. 2, pp. 309–319, 2010.
[254] D. Dong and I. R. Petersen, "Notes on sliding mode control of two-level quantum systems," *Automatica*, vol. 48, no. 12, pp. 3089–3097, 2012.
[255] C. D'Helon, A. Doherty, M. James, and S. Wilson, "Quantum risk-sensitive control," in *Proceedings of the 45th IEEE Conference on Decision and Control*. IEEE, 2006, pp. 3132–3137.
[256] M. James, "A quantum langevin formulation of risk-sensitive optimal control," *Journal of Optics B: Quantum and Semiclassical Optics*, vol. 7, no. 10, p. S198, 2005.
[257] P. Dupuis and R. S. Ellis, *A weak convergence approach to the theory of large deviations*. John Wiley & Sons, 2011, vol. 902.
[258] N. Yamamoto, "Robust observer for uncertain linear quantum systems," *Physical Review A*, vol. 74, no. 3, p. 032107, 2006.
[259] Z. Leghtas, A. Sarlette, and P. Rouchon, "Adiabatic passage and ensemble control of quantum systems," *Journal of Physics B: Atomic, Molecular and Optical Physics*, vol. 44, no. 15, p. 154017, 2011.
[260] J.-S. Li and N. Khaneja, "Control of inhomogeneous quantum ensembles," *Physical review A*, vol. 73, no. 3, p. 030302, 2006.
[261] M. H. Levitt, "Composite pulses," *Progress in Nuclear Magnetic Resonance Spectroscopy*, vol. 18, no. 2, pp. 61–122, 1986.
[262] Z.-C. Shi, H.-N. Wu, L.-T. Shen, J. Song, Y. Xia, X. Yi, and S.-B. Zheng, "Robust single-qubit gates by composite pulses in three-level systems," *Physical Review A*, vol. 103, no. 5, p. 052612, 2021.
[263] N. Augier, U. Boscain, and M. Sigalotti, "Adiabatic ensemble control of a continuum of quantum systems," *SIAM Journal on Control and Optimization*, vol. 56, no. 6, pp. 4045–4068, 2018.
[264] C. Chen, D. Dong, B. Qi, I. R. Petersen, and H. Rabitz, "Quantum ensemble classification: A sampling-based learning control approach," *IEEE transactions on neural networks and learning systems*, vol. 28, no. 6, pp. 1345–1359, 2016.
[265] Y. Sun, H. Ma, C. Wu, C. Chen, and D. Dong, "Ensemble control of open quantum systems using differential evolution," in *2015 10th Asian Control Conference (ASCC)*. IEEE, 2015, pp. 1–6.
[266] D. Dong, X. Xing, H. Ma, C. Chen, Z. Liu, and H. Rabitz, "Learning-based quantum robust control: algorithm, applications, and experiments," *IEEE transactions on cybernetics*, vol. 50, no. 8, pp. 3581–3593, 2019.
[267] Y. Zhang, "General robust-optimization formulation for nonlinear programming," *Journal of optimization theory and applications*, vol. 132, no. 1, pp. 111–124, 2007.
[268] D. Bertsimas, O. Nohadani, and K. M. Teo, "Nonconvex robust optimization for problems with constraints," *INFORMS journal on comput-*

ing, vol. 22, no. 1, pp. 44–58, 2010.

[269] ——, "Robust optimization for unconstrained simulation-based problems," *Operations research*, vol. 58, no. 1, pp. 161–178, 2010.

[270] A. Mutapcic and S. Boyd, "Cutting-set methods for robust convex optimization with pessimizing oracles," *Optimization Methods & Software*, vol. 24, no. 3, pp. 381–406, 2009.

[271] A. Mutapcic, S. Boyd, A. Farjadpour, S. G. Johnson, and Y. Avniel, "Robust design of slow-light tapers in periodic waveguides," *Engineering Optimization*, vol. 41, no. 4, pp. 365–384, 2009.

[272] A. Oskooi, A. Mutapcic, S. Noda, J. D. Joannopoulos, S. P. Boyd, and S. G. Johnson, "Robust optimization of adiabatic tapers for coupling to slow-light photonic-crystal waveguides," *Optics express*, vol. 20, no. 19, pp. 21 558–21 575, 2012.

[273] J. R. West, B. H. Fong, and D. A. Lidar, "Near-optimal dynamical decoupling of a qubit," *Physical review letters*, vol. 104, no. 13, p. 130501, 2010.

[274] H. Y. Carr and E. M. Purcell, "Effects of diffusion on free precession in nuclear magnetic resonance experiments," *Physical review*, vol. 94, no. 3, p. 630, 1954.

[275] G. S. Uhrig, "Keeping a quantum bit alive by optimized π-pulse sequences," *Physical Review Letters*, vol. 98, no. 10, p. 100504, 2007.

[276] K. Khodjasteh and D. A. Lidar, "Fault-tolerant quantum dynamical decoupling," *Physical review letters*, vol. 95, no. 18, p. 180501, 2005.

[277] A. M. Souza, G. A. Alvarez, and D. Suter, "Robust dynamical decoupling for quantum computing and quantum memory," *Physical review letters*, vol. 106, no. 24, p. 240501, 2011.

[278] K. Khodjasteh, D. A. Lidar, and L. Viola, "Arbitrarily accurate dynamical control in open quantum systems," *Physical review letters*, vol. 104, no. 9, p. 090501, 2010.

[279] M. J. Biercuk, H. Uys, A. P. VanDevender, N. Shiga, W. M. Itano, and J. J. Bollinger, "Optimized dynamical decoupling in a model quantum memory," *Nature*, vol. 458, no. 7241, pp. 996–1000, 2009.

[280] B. Pokharel, N. Anand, B. Fortman, and D. A. Lidar, "Demonstration of fidelity improvement using dynamical decoupling with superconducting qubits," *Physical review letters*, vol. 121, no. 22, p. 220502, 2018.

[281] P. Zhao, X. Wu, and D. Tong, "Dynamical-decoupling-protected nonadiabatic holonomic quantum computation," *Physical Review A*, vol. 103, no. 1, p. 012205, 2021.

[282] G. A. Alvarez, A. Ajoy, X. Peng, and D. Suter, "Performance comparison of dynamical decoupling sequences for a qubit in a rapidly fluctuating spin bath," *Physical Review A*, vol. 82, no. 4, p. 042306, 2010.

[283] G. A. Paz-Silva, S.-W. Lee, T. J. Green, and L. Viola, "Dynamical decoupling sequences for multi-qubit dephasing suppression and long-time quantum memory," *New Journal of Physics*, vol. 18, no. 7, p. 073020, 2016.

[284] C. A. Ryan, J. S. Hodges, and D. G. Cory, "Robust decoupling techniques to extend quantum coherence in diamond," *Physical Review Letters*, vol. 105, no. 20, p. 200402, 2010.

[285] T. E. Hodgson, L. Viola, and I. D'Amico, "Towards optimized suppression of dephasing in systems subject to pulse timing constraints," *Physical Review A*, vol. 81, no. 6, p. 062321, 2010.

[286] A. M. Souza, G. A. Álvarez, and D. Suter, "Robust dynamical decoupling," *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, vol. 370, no. 1976, pp. 4748–4769, 2012.

[287] T. Gullion, D. B. Baker, and M. S. Conradi, "New, compensated carr-purcell sequences," *Journal of Magnetic Resonance (1969)*, vol. 89, no. 3, pp. 479–484, 1990.

[288] H. K. Ng, D. A. Lidar, and J. Preskill, "Combining dynamical decoupling with fault-tolerant quantum computation," *Physical Review A*, vol. 84, no. 1, p. 012305, 2011.

[289] G. A. Álvarez and D. Suter, "Measuring the spectrum of colored noise by dynamical decoupling," *Physical review letters*, vol. 107, no. 23, p. 230501, 2011.

[290] I. Almog, Y. Sagi, G. Gordon, G. Bensky, G. Kurizki, and N. Davidson, "Direct measurement of the system–environment coupling as a tool for understanding decoherence and dynamical decoupling," *Journal of Physics B: Atomic, Molecular and Optical Physics*, vol. 44, no. 15, p. 154006, 2011.

[291] J. Taylor, P. Cappellaro, L. Childress, L. Jiang, D. Budker, P. Hemmer, A. Yacoby, R. Walsworth, and M. Lukin, "High-sensitivity diamond magnetometer with nanoscale resolution," *Nature Physics*, vol. 4, no. 10, pp. 810–816, 2008.

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