Advanced Journal of Graduate Research
ISSN: 2456-7108
Volume 13, Issue 1, pp. 37-61, January 2023
doi:https://doi.org/10.21467/ajgr.13.1.37-61
GRADUATE RESEARCH ARTICLE

Solution Methods for Nonlinear Ordinary Differential Equations Using Lie Symmetry Groups

Disanayakage Hashan Sanjaya Perera*, Dilruk Gallage
University of Colombo, Sri Lanka

***Corresponding Author:**
hashanperera444@gmail.com

Article History
Received: 29 July 2022
Revised: 07 February 2023
Accepted: 07 February 2023
Published: 28 February 2023

Student

- Disanayakage Hashan Sanjaya Perera

Academic Year: 2020 - 2021
Course Level: Bachelor
Course Name: Mathematical Finance
Course Year: Final Year

Mentor

- Dr. P D Dilruk Gallage

ABSTRACT

For formulating mathematical models, engineering problems and physical problems, Nonlinear ordinary differential equations(NODEs) are used widely. Nevertheless, explicit solutions can be obtained for very few NODEs, due to lack of techniques to obtain explicit solutions. Therefore methods to obtain approximate solution for NODEs are used widely. Although symmetry groups of ordinary differential equations (ODEs) can be used to obtain exact solutions however, these techniques are not widely used. The purpose of this paper is to present applications of Lie symmetry groups to obtain exact solutions of NODEs . In this paper we connect different methods,theorems and definitions of Lie symmetry groups from different references and we solve first order and second order NODEs. In this analysis three different methods are used to obtain exact solutions of NODEs. Using applications of these symmetry methods, drawbacks and advantages of these different symmetry methods are discussed and some examples have been illustrated graphically. Focus is first placed on discussing about the notion of symmetry groups of the ODEs. Focus is then changed to apply them to find general solutions for NODEs under three different methods. First we find suitable change of variables that convert given first order NODE into variable separable form using these symmetry groups. As another method to find general solutions for first order NODEs, we find particular type of solution curves called invariant solution curves under Lie symmetry groups and we use these invariant solution curves to obtain the general solutions. We find general solutions for the second order NODEs by reducing their order to first order using Lie symmetry groups.

Keywords: differential equations, nonlinear, Lie symmetry groups

1 Introduction

Lie group analysis is used in many different fields such as control theory, algebraic topology,differential geometry,relativity [1], physics [2, 3, 4, 5, 6, 7, 8], geometry, mechanics [9], Chemistry and Chemical biology [10]. Moreover it has been used in mathematical models such as models regarding deceases [11, 12, 13], models of epidemics [14] and finance [15, 16].

Although there are many different standard methods to obtain general solutions of ODEs, these techniques can be used only for specific types of ordinary differential equations (ODEs), such as linear, separable, homogeneous, ecaxt...etc. Therefore methods for finding approximations to the solutions are widely used. Maris Marius Sophus Lie who was a Norwegian mathematician, discovered that most of these standard solution methods are based on the symmetry groups of ODEs [1] and in the 1880s he introduced the concept of continuous symmetry groups called Lie symmetry groups [17] which can be used to obtain exact solutions of ODEs including which are not fit into standard type that we mentioned above.

For formulating mathematical models, engineering problems and physical problems, Nonlinear ordinary differential equations (NODEs) are used widely [18][2]. Nevertheless, explicit solutions can be obtained for very few NODEs [19]. Most of the standard solution methods are insufficient to obtain exact solutions of NODEs [19]. Although symmetry methods can be used to obtain the exact solutions of ODEs, these methods are not widely used. In this thesis we present basic theories and definitions regarding Lie symmetry groups and we use these symmetry groups to obtain exact solutions of first and second order NODEs. For the first order NODEs two different methods are presented to obtain exact solutions. Mainly, the Lie symmetry groups of a given NODE are used to find suitable change of variables that convert the NODE into variable separable form. Then we obtain exact solution by integration. In the second method we find special type of solution curves of a given NOD called invariant solution curves under Lie symmetry groups and we map these invariant solution curves to other solution curves using different Lie symmetry groups, which leads to obtain general solution of the NODE. For the higher order NODEs Lie symmetry groups are used to find appropriate change of variables which can be used to reduce the order of the NODEs. For this purpose we consider extension of symmetry groups in two dimensional euclidean space to higher dimensional euclidean space.

2 Symmetries and One Parameter Lie Group of Transformations.

For understanding the notion of symmetries of ODEs, it is supportive to understand the symmetries of geometrical planer objects. A symmetry of a geometrical planer object is a transformation whose action does not change the physical appearance of the object [20]. As an example, for an equilateral triangle , anti-clock wise or clock-wise rotations of $\frac{4\pi}{3}$,2π and $\frac{2\pi}{3}$ about the center are symmetries , since the first object and its image under these transformations are indistinguishable. If a symmetry of a mathematical object can't be defined by a continuous parameter, that symmetry is called discrete symmetry [20]. Therefore above rotations can be considered as discrete symmetries of the equilateral triangle. Consider a unit circle given by $cos(\theta)^2 + sin(\theta)^2 = 1$. Then a rotation of $\varepsilon \in (-\pi, \pi]$ angel around the center is a symmetry and it can be represented by $cos(\theta+\varepsilon)^2 + sin(\theta+\varepsilon)^2 = 1$. This transformation can be considered as a continuous symmetry since ε can be varied continuously i.e $\varepsilon \in (-\pi, \pi]$. To discuss about symmetries of ODEs we restrict our attention to continuous symmetries. Symmetries of ODEs are defined under one parameter Lie group of transformations. Next definition of one parameter Lie group of transformation is interpreted from [21]

Definition 2.1 (One parameter Lie group of transformations). *Let $v = (x_1, ..., x_n)$,$v \in U, U \subset \mathbb{R}^n$. The set of transformations*

$$v^* = \Psi(v, \varepsilon)$$

$\forall v \in U$, $\varepsilon \in Q$,$Q \subset \mathbb{R}$, *represent one parameter Lie group of transformations(OLGT) on U if it satisfies following seven conditions.* $\phi(\varepsilon, \delta)$ *represents the composition of parameters* $\delta, \varepsilon \in Q$

1. ε ; $\varepsilon \in Q$ the transformations are bijective transformations in U.
2. Q and ϕ form a group G and $\varepsilon = 0$ corresponds the the identity element e of G.
3. Q is a sub interval of R.
4. $\Psi(v, 0) = v$
5. If $v^{**} = \Psi(v^*, \delta)$ then $v^{**} = \Psi(v, \phi(\varepsilon, \delta))$
6. ψ is infinitely differentible with respect to $v \in U$ and analytic function of $\varepsilon \in Q$.
7. $\phi(\varepsilon, \delta)$ is an analytic function of δ and ε.

These point transformations have been defined depending on one continuous parameter ε. The group defined in condition 2 is called **Local Lie group** [1]. In this paper following notations are used to represent One parameter Lie group of transformations acting on space $\mathbb{R}^2$.

$$v^* = (x^*, y^*) = \Psi(v, \varepsilon) = (X(x, y; \varepsilon), Y(x, y; \varepsilon)) = (X(v; \varepsilon), Y(v; \varepsilon)) \quad (1)$$

Perera & Gallage, *Adv. J. Grad. Res.*; Vol. 13, Issue 1, pp: 37-61, January 2023

where $v = (x, y)$, X and Y represents smooth functions and ε is a real parameter.

A OLGT is a symmetry group(Lie symmetry group) of a ODE if the transformations map any solution curve into another solution curve [1, 21]. A Lie group of transformations is said to be **admitted** by an ODE if it is a Lie symmetry group of that ODE [21].

Let 1 be a one parameter Lie group of transformations admitted by an ODE. Then this group of transformations map any solution curve of the ODE into another solution curve. Therefore a family of solution curves of the ODE and its image under the transformations are indistinguishable. Then that one parameter Lie group of transformations satisfies the symmetry condition of the family of solution curves called the symmetry condition of the ODE. Lie symmetry groups are manifolds and their forward transformations and inverse transformations are diffeomorphisms [22]. Using these properties we can prove that these transformations satisfy all conditions to be a symmetry of an ODE [23]. Mathematical expression for the symmetry condition of ODEs is given in the next definition interpreted from [20]

Definition 2.2. Consider the n^{th} order ODE given by $\frac{d^n y}{dx^n} = g(x, y, \underset{1}{y}, ..., \underset{n-1}{y})$. Let 1 be a OLGT admitted by $g(x, y, \underset{1}{y}, ..., \underset{n-1}{y})$. Then the symmetry condition of $g(x, y, \underset{1}{y}, ..., \underset{n-1}{y})$ is given by

$$\frac{d^n y^*}{dx^{*n}} = g(x^*, y^*, \underset{1}{y^*}, ..., \underset{n-1}{y^*}) \;\; when \;\; \frac{d^n y}{dx^n} = g(x, y, \underset{1}{y}, ..., \underset{n-1}{y}). \tag{2}$$

where $\underset{t}{y} = \frac{d^t y}{dx^t}$ and $\underset{q}{y^*} = \frac{d^q y^*}{dx^{*t}}$, t = 1,2,3,...,n

The condition 2 implies that family of solution curves in (x, y)-plane and its image under the LOGT in (x^*, y^*)-plane are indistinguishable.

Definition 2.3 (Orbit)**.** Consider the one parameter Lie group of transformations 1. The orbit of $v^* = \Psi(v, \varepsilon)$ passing through point v is the set of all points that v can be mapped by the appropriate selection of ε.[20]

Consider $v^* = \Psi(v, \varepsilon)$ as OLGT admitted by a ODE . Then $v^* = \Psi(v, \varepsilon)$ continuously maps one solution curve into other solution curves of the ODE. Consider a one particular point v on a solution curve. Then the path that point v is mapped into other points in other solution curves can be considered as an orbit. Following definition is interpreted from [21] .

Definition 2.4 (Infinitesimals)**.** Let $v = (x_1, x_2, ..., x_4) \in Q$, $Q \subset \mathbb{R}$. Consider the OLGT $\Psi(v, \varepsilon)$ defined in definition 2.1.

$$\Theta(v) = \left.\frac{\partial \Psi(v, \varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0} \tag{3}$$

$\Theta(v)$ are called the infinitesimals of Lie group of transformations.

The infinitesimals of One parameter Lie group of transformations 1 acting on $\mathbb{R}^2$ space are represented by the standard notations $\xi(x, y)$ and $\eta(x, y)$ as follows .

$$\begin{aligned}\Theta(v) &= \left(\left.\frac{dx^*}{d\varepsilon}\right|_{\varepsilon=0}, \left.\frac{dy^*}{d\varepsilon}\right|_{\varepsilon=0} \right) \\ &= (\xi(x, y), \eta(x, y))\end{aligned}$$

$\Theta(v)$ represents tangent vector field of the corresponding OLGT [20]. The maximal integral curves (flow curves) generated by vector field $\Theta(v)$ same as the corresponding one parameter Lie group of transformations with the same domain and the infinitely many orbits of $v^* = \Psi(v, \varepsilon)$ represents these maximal integral curves [1]. For the justification of above last statement refer [1] from pages 24 to 28.

Example 2.5. Consider The OLGT $v^* = \psi(v, \varepsilon)$ given by $x^* = e^{\varepsilon} x$ and $y^* = e^{-\varepsilon} y$ [17] where $v^* = (x^*, y^*)$,$v = (x, y)$ and ε is a real parameter. Then the infinitesimals of $v^* = \psi(v, \varepsilon)$ is given by $(\frac{dx^*}{d\varepsilon}|_{\varepsilon=0}, \frac{dy^*}{d\varepsilon}|_{\varepsilon=0}) =$

$(x, -y)$. The vector field $(x, -y)$ and some flow curves generated by the vector field are illustrated in Figure 1. The black color curves represent the flow curves or the orbits.

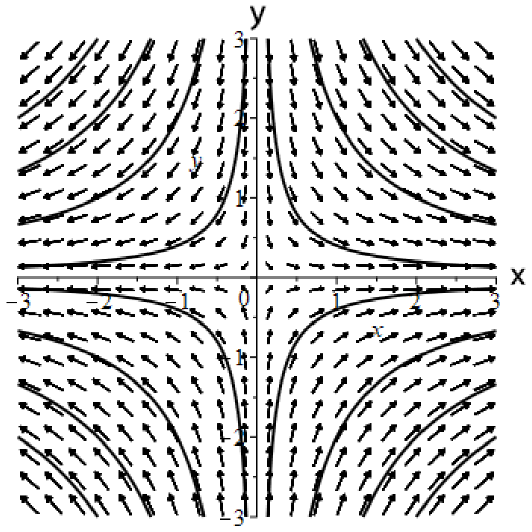

Figure 1: vector field (x,-y) with flow curves

Definition 2.6 (Infinitesimal Generator)**.** Let $v = (x_1, ..., x_n) \in Q$, $Q \subset \mathbb{R}$.Consider the OLGT defined in definition 2.1. The infinitesimal generator of $v^* = \Psi(v, \varepsilon)$ is given by

$$\bar{X} = \bar{X}(v) = \sum_{i=1}^{n} \Theta_i(v) \frac{\partial}{\partial v_i} \tag{4}$$

For any differentiable function $g(v) = g(x_1, x_2, ..., x_n)$

$$\bar{X} g(v) = \sum_{i=1}^{n} \Theta_i(v) \frac{\partial g(\mathbf{v})}{\partial v_i}$$

Above definition has been interpreted from [21]. The infinitesimal generator is used to define the Lie algebra which is used for finding solutions for partial differential equations [2, 3, 4, 7] and for symmetry generating models [24].

Theorem 2.7 (First fundamental theorem of Lie)**.** *There exists a parameterization $\tau(\varepsilon)$ such that OLGT defined in definition 2.1 is equivalent to the solution of the initial value problem*

$$\frac{dv^*}{d\Upsilon} = \Theta(v^*) \tag{5}$$

with $v^* = v$ when $\tau = 0$. where

$$\Upsilon(\varepsilon) = \int_0^{\varepsilon} \omega(\varepsilon)\, d\varepsilon$$

$$\omega(\varepsilon) = \left. \frac{\partial \phi(\mu, \lambda)}{\partial \lambda} \right|_{(\mu,\lambda)=(\varepsilon,\varepsilon^{-1})}$$

$\omega(0) = 1$, *μ, λ are real parameters and ε^{-1} is the inverse element of ε in the local Lie group G. Under this re-parameterization composition of parameters becomes additive.*

For the proof refer [21] page 38. We can use first fundamental theorem of Lie to obtain the Lie symmetry groups of a given ordinary differential equation when the corresponding infinitesimals are known.

Consider the one parameter Lie group of transformations 1 with the infinitesimals $\vartheta(v) = (\xi(x,y), \eta(x,y))$. According to first fundamental theorem of Lie we can consider $v^* = \Psi(v, \varepsilon)$ as one parameter Lie group of transformations where the composition of parameters is additive. Let μ and λ be two parameters and ϕ be the composition of parameters. Then

$$\frac{\partial \phi(\mu,\lambda)}{\partial \lambda} = \frac{\partial(\mu+\lambda)}{\partial \lambda} = 1$$

Hence

$$\omega(\varepsilon) = \frac{\partial \phi(\mu,\lambda)}{\partial \lambda}\bigg|_{(\mu,\lambda)=(\varepsilon,\varepsilon^{-1})} = 1\bigg|_{(\mu,\lambda)=(\varepsilon,\varepsilon^{-1})} = 1$$

Then

$$\begin{aligned}
\Upsilon(\varepsilon) &= \int_0^{\varepsilon} 1 d\varepsilon = \varepsilon \\
\frac{dv^*}{d\Upsilon(\varepsilon)} &= \Theta(v^*) \\
\frac{dv^*}{d\varepsilon} &= \Theta(v^*)
\end{aligned}$$

where $v^* = v$ when $\varepsilon = 0$.Therefore the initial value problem 5 becomes

$$\frac{dx^*}{d\varepsilon} = \xi(x^*, y^*) \qquad \frac{dy^*}{d\varepsilon} = \eta(x^*, y^*) \tag{6}$$

with $x^* = x, y^* = y$ when $\varepsilon = 0$.

Example 2.8. For this example a NODE is taken from the exercises of [20]. Consider the NODE $\frac{dy}{dx} = \frac{3y}{x} + \frac{x^5}{2y+x^3}$. $(\xi(x,y), \eta(x,y))=(x, 3y)$ [20] are infinitesimals of a Lie symmetry group of the NODE. Let $v^* = \Psi(v, \varepsilon)$ be a one parameter Lie group of transformations with the given infinitesimals where $v^* = (x^*, y^*), v = (x, y)$ and ε is a real parameter. By 6

$$\frac{dx^*}{d\varepsilon} = \xi(x^*, y^*) \qquad \frac{dy^*}{d\varepsilon} = \eta(x^*, y^*)$$

with $x^* = x, y^* = y$ when $\varepsilon = 0$. Then to obtain x^* and y^* we solve above two ODEs respectively.

$$\begin{aligned}
\frac{dx^*}{d\varepsilon} &= \xi(x^*, y^*) = x^* \\
\frac{dx^*}{x^*} &= d\varepsilon \\
\int \frac{dx^*}{x^*} &= \int d\varepsilon \\
\ln(x^*) + c(x,y) &= \varepsilon
\end{aligned}$$

$v^* = v$ when $\varepsilon = 0$.Hence $x^* = x$ when $\varepsilon = 0$. Therefore

$$\begin{aligned}
\ln(x) + c(x,y) &= 0 \\
c(x,y) &= -\ln(x)
\end{aligned}$$

Then

$$\begin{aligned}
\ln(x^*) - \ln(x) &= \varepsilon \\
\ln(\frac{x^*}{x}) &= \varepsilon \\
x^* &= e^{\varepsilon} x
\end{aligned}$$

Similarly we can obtain $y^* = e^{3\varepsilon}y$ by solving the initial value problem $\frac{dy^*}{d\varepsilon} = \eta(x^*, y^*) = 3y^*$with $y^* = y$ when $\varepsilon = 0$. Therefore $v^* = \Psi(v, \varepsilon) = (e^{\varepsilon}x, e^{3\varepsilon}y)$ and composition of parameters is additive. We can check whether this is a symmetry group or not for the given ODE by checking the symmetry condition of the given ODE.

$$\frac{dy}{dx} = \frac{3y}{x} + \frac{x^5}{2y + x^3}$$

we can obtain $\frac{dy^*}{dx^*}$ in terms of x and y by the chain rule.

$$\begin{aligned}\frac{dy^*}{dx^*} &= \frac{dy^*}{dx} / \frac{dx^*}{dx} \\ &= e^{2\varepsilon}\frac{dy}{dx}\end{aligned}$$

Then we can obtain $\frac{3y^*}{x^*} + \frac{x^{*5}}{2y^* + x^{*3}}$ in terms of x and y by substituting the transformations.

$$\begin{aligned}\frac{3y^*}{x^*} + \frac{x^{*5}}{2y^* + x^{*3}} &= \frac{3e^{3\varepsilon}y}{e^{\varepsilon}x} + \frac{e^{5\varepsilon}x^5}{2e^{3\varepsilon}y + e^{3\varepsilon}x} \\ &= e^{2\varepsilon}\frac{dy}{dx}\end{aligned}$$

Therefore we can obtain

$$\frac{dy^*}{dx^*} = \frac{3y^*}{x^*} + \frac{x^{*5}}{2y^* + x^{*3}}$$

Therefore $\frac{dy^*}{dx^*} = \frac{3y^*}{x^*} + \frac{x^{*5}}{2y^* + x^{*3}}$ when $\frac{dy}{dx} = \frac{3y}{x} + \frac{x^5}{2y + x^3}$. Hence $v^* = \Psi(v, \varepsilon)$ satisfies the symmetry condition for the given ODE.

Definition 2.9 (Total derivative Operator)**.** The total derivative with respect to x is given by

$$D_x = \frac{\partial}{\partial x} + \underset{1}{y}\frac{\partial}{\partial y} + ... + \underset{n+1}{y}\frac{\partial}{\partial \underset{n}{y}} + ... \tag{7}$$

where $\frac{dy}{dx} = \frac{D_x y}{D_x x}$ and $\underset{n}{y} = \frac{d^n y}{dx^n}$, n=1,2,3...[21]

2.1 Canonical Coordinates

If a given ODE exits in variable separable form we can obtain the exact solution by the solution method separation of variables. Otherwise we can convert the ODE into separable form using appropriate change of variables. But finding this appropriate change of variables is not always easy task. This section contains method of finding appropriate change of variable that convert a given ODE into separable form using its Lie symmetry groups.

Definition 2.10. [Canonical Coordinates] Let $v = (x_1, x_2, ..., x_n)$ lie in region $D \subset \mathbb{R}^n$and $v^* = \psi(v, \varepsilon)$ be a one parameter Lie group of transformations defined in 2.1. Consider the change of coordinates (one to one and continuously differentiable in the domain) $u = (y_1(x), y_2(x), ..., y_n(x))$.Then $\bar{X} = \bar{Y}$ .Where $\bar{X}$ is the infinitesimal generator of $v^* = \Psi(v, \varepsilon)$ in terms of coordinates $\mathbf{x}$ and $\bar{Y}$ is infinitesimal generator of $v^* = \Psi(v, \varepsilon)$ in terms of $\mathbf{y}$.

Then u is called a set of canonical coordinates for the Lie symmetry $v^* = \Psi(v, \varepsilon)$ if with respect to u the one parameter Lie group of transformations becomes

$$y_i^* = y_i, \quad i = 1, 2, ..., n-1$$

$$y_n^* = y_n + \varepsilon$$

Above definition have been interpreted based on [21]. One parameter Lie group of transformations under the canonical coordinates is called **canonical form** of the one parameter Lie group of transformations [25].

Consider the first order ordinary differential equation give by $\frac{dy}{dx} = f(x,y)$. Let 1 be a Lie symmetry group of the ODE. Let $(r(x,y), s(x,y))$ be the canonical coordinates of $v^* = \psi(v,\varepsilon)$ and $\frac{ds}{dr} = F(r,s)$ be the given ODE in terms of Canonical coordinates. Then according to definition 2.10 in terms of these canonical coordinates $v^* = \psi(v,\varepsilon)$ becomes its canonical form given by

$$r^* = r \qquad s^* = s + \varepsilon$$

Above the canonical form of v^* implies that transformations map one solution curve of $F(r,s)$ to another solution curve into direction of dependent variable s. It means that the slop of the solutions curves does not depend on variable s. Hence $F(r,s)$ exits in variable separable form. we can justify that $F(r,s)$ is separable as follows.

By the symmetry condition of $\frac{ds}{dr} = F(r,s)$ we can obtain

$$\frac{ds^*}{dr^*} = F(r^*, s^*) = F(r, s+\varepsilon)$$

Using total derivative operator

$$\begin{aligned}\frac{ds^*}{dr^*} &= \frac{s_r^* + s_s^*.s'}{r_r^* + r_s^*.s'}\\ &= \frac{ds}{dr}\\ &= F(r,s)\end{aligned}$$

Therefore $F(r, s+\varepsilon) = F(r,s)$. Hence $F(r,s)$ does not depend on variable s. Then the ODE $\frac{ds}{dr} = F(r,s)$ can be written in the form $\frac{ds}{dr} = F(r)$. Hence $\frac{ds}{dr}$ exists in variable separable form.

Let $\bar{X}$ be the infinitesimal generator of $v^* = \Psi(v,\varepsilon)$ with respect to coordinates $v = (x,y)$ and $\bar{Y}$ be the infinitesimal generator of $v^* = \Psi(v,\varepsilon)$ with respect to canonical coordinates $u = (r(x,y), s(x,y))$. The infinitesimals of $v^* = \Psi(v,\varepsilon)$ with respect to coordinates u are given by

$$\left.\frac{\partial r^*}{\partial \varepsilon}\right|_{\varepsilon=0} = 0 \quad \left.\frac{\partial s^*}{\partial \varepsilon}\right|_{\varepsilon=0} = 1$$

Therefore the infinitesimal generator $\bar{Y}$ can be written in the form

$$\bar{Y} = \frac{\partial}{\partial s}$$

The infinitesimals of $v^* = \Psi(v,\varepsilon)$ with respect to coordinates v are given by

$$\xi(x,y) = \left.\frac{\partial x^*}{\partial \varepsilon}\right|_{\varepsilon=0} \qquad \eta(x,y) = \left.\frac{\partial y^*}{\partial \varepsilon}\right|_{\varepsilon=0}$$

Then $\bar{X}$ can be written in the form

$$\bar{X} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$$

By definition 2.10

$$\bar{X}r = \xi(x,y)\frac{\partial r}{\partial x} + \eta(x,y)\frac{\partial r}{\partial y} = \bar{Y}r = 0$$

$$\bar{X}s = \xi(x,y)\frac{\partial s}{\partial x} + \eta(x,y)\frac{\partial s}{\partial y} = \bar{Y}s = 1$$

By solving following two partial differential equations using **method of characteristics** [26][2] we can find the canonical coordinates $(r(x,y), s(x,y))$.

$$\xi(x,y)\frac{\partial r}{\partial x} + \eta(x,y)\frac{\partial r}{\partial y} = 0 \tag{8}$$

$$\xi(x,y)\frac{\partial s}{\partial x}+\eta(x,y)\frac{\partial s}{\partial y}=1 \tag{9}$$

Above construction of equations 8 and 9 has been interpreted based on the content of [21].

Theorem 2.11. *Every one parameter Lie group of transformations can be reduced into canonical form[25]*

By theorem 2.11 we can conclude that for any one parameter Lie group of transformations there is canonical coordinates which reduce it into canonical form.

2.2 Invariant Curves

For some ODEs exits solution curves of a given ODE that can't be mapped into another solution curves by a Lie symmetry group of that ODE because of the transformations map these curves to itself. These curves are called invariant solution curves under the Lie group of transformations [21]. Following theorem is interpreted from [21]

Theorem 2.12. *[Invarient Curves] A curve written on solved form G(x,y)= y - g(x)=0 is an invariant curve for a one parameter Lie group of transformations* $v^*=\psi(v,\varepsilon)$ *whose tangent vector field is* $(\xi(x,y),\eta(x,y))$. *with infinitesimal generator* $\boldsymbol{X}=\xi(x,y)\frac{\partial}{\partial x}+\eta(x,y)\frac{\partial}{\partial y}$ *if and only if*

$$\boldsymbol{X}G(x,y)=\eta(x,y)-\xi(x,y)g'(x)=0 \tag{10}$$

If a solution curve of a ODE satisfies 10 under a Lie symmetry group of the ODE we called it **invariant solution curve** of the ODE under the Lie symmetry group. A curve is invariant under a one parameter Lie group of transformations if and only if no orbit crosses that curve [20]. Therefor invariant solution curves of a ODE under a Lie symmetry group are not crossed by the infinitely many orbits or infinitely many flow curves generated by the vector field $(\xi(x,y),\eta(x,y))$.

Example 2.13. Consider the Riccati-type ordinary differential equation $\frac{dy}{dx}=y^2-\frac{y}{x}-\frac{1}{x^2}$. Its Lie symmetry group is given by $x^*=e^{\varepsilon}x, y^*=e^{-\varepsilon}y$ [17]. Then infinitesimals are given by $(\xi(x,y),\eta(x,y))=(x,-y)$. Using 10 we can obtain the invariant solution curves under the corresponding Lie symmetry group.

$$\eta(x,y)-\xi(x,y)(\frac{dy}{dx})=0$$

$$-y-x(y^2-\frac{y}{x}-\frac{1}{x^2})=0$$

$$y=\frac{1}{x} \quad y=\frac{-1}{x} \tag{11}$$

Then under the transformations $x^*=e^{\varepsilon}x \quad ,y^*=e^{-\varepsilon}y$ the invariant curve $y=\frac{1}{x}$ becomes

$$y^*=\frac{1}{x^*}$$

Then

$$e^{-\varepsilon}y=\frac{1}{e^{\varepsilon}x}$$

$$y=\frac{1}{x}$$

Therefore transformations in the symmetry group map the invariant curve $y=\frac{1}{x}$ to itself.

In Figure 1 we have presented the vector field $(x,-y)$ and some flow curves. Figure 2 illustrates how these invariant curves exist in the vector field $(x,-y)$ with the flow curves illustrated in Figure 1. Blue curves represent the invariant curve $y_2=\frac{-1}{x}$ and red curves represent the invariant curve $y_1=\frac{1}{x}$.

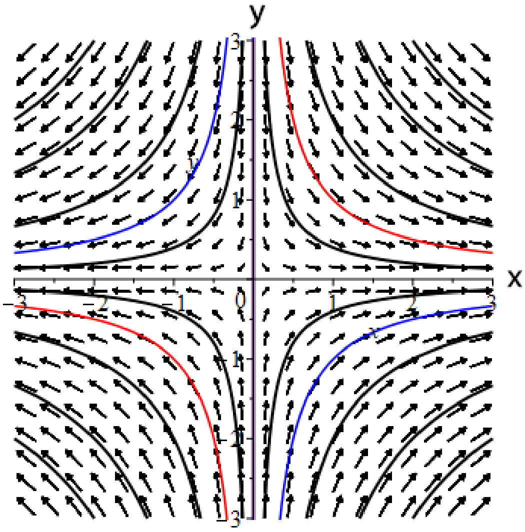

Figure 2: vector field (x,-y) with flow curves and curves $\frac{1}{x}$, $\frac{-1}{x}$

Figure 2 shows how these invariant solution curves exist in a way that no flow curve crosses these curves. In some physical phenomena, the invariant solution curves can be used to explain the important features [4] but in this project, We use invariant solution curves under a Lie symmetry group to obtain the general solution of the corresponding ODE. Two solved examples have been given in chapter 6.

3 Prolongation of One Parameter Lie Group of Transformations

Prolongation of Lie group of transformations is the extension of a Lie group of transformations defined on two dimensional euclidean space to a Lie group of transformations defined on $k+1(k>1)$ dimensional euclidean space. In this chapter we consider x as the only independent variable and other variables are dependent variables . Next theorem is interpreted from [21]

Theorem 3.1. *Let $v^* = (x^*, y^*)$ be a one parameter Lie group of transformations admitted by k^{th} $(k > 1)$order ordinary differential equation where $v^* = (x^*, y^*), v = (x, y)$,$\varepsilon$ is a real parameter and*

$$x^* = X(x, y; \varepsilon) = X(v; \varepsilon) \quad y^* = Y(x, y; \varepsilon) = Y(v; \varepsilon) \tag{12}$$

X and Y are smooth functions. Let $\underset{q}{y} = \frac{d^q y}{dx^q}$ and $\underset{q}{y^*} = \frac{d^q y^*}{dx^{*q}}$, q = 1,2,3,... .Then

$$\underset{1}{y^*} = Y^1(x, y, \underset{1}{y}; \varepsilon) = \frac{\frac{\partial Y(v;\varepsilon)}{\partial x} + \frac{\partial Y(v;\varepsilon)}{\partial y}\underset{1}{y}}{\frac{\partial X(v;\varepsilon)}{\partial x} + \frac{\partial X(v;\varepsilon)}{\partial y}\underset{1}{y}} \tag{13}$$

This theorem shows that the first derivative of transformed dependent variable y^* can be represented by parameterized transformation $Y^1(x, y, \underset{1}{y}; \varepsilon)$.

Theorem 3.2. *The Lie group of transformations 12 acting on (x,y)-space extends to its k^{th} extension, which is the following one parameter Lie group of transformation acting on $(x, y, \underset{1}{y}, \underset{2}{y}, ..., \underset{k}{y})$- space*

$$x^* = X(x, y; \varepsilon)$$

$$y^* = Y(x, y; \varepsilon)$$

$$\underset{1}{y}^* = Y^1(x, y, \underset{1}{y}; \varepsilon)$$

$$\vdots$$

$$\underset{k}{y}^* = Y^k(x, y, \underset{1}{y}, \underset{2}{y}, \cdots, \underset{k}{y}; \varepsilon) = \frac{\frac{\partial Y^{k-1}}{\partial x} + \underset{1}{y}.\frac{\partial Y^{k-1}}{\partial y} + \cdots + \underset{k}{y}.\frac{\partial Y^{k-1}}{\partial \underset{k-1}{y}}}{\frac{\partial X(v;\varepsilon)}{\partial x} + \underset{1}{y}.\frac{\partial X(v;\varepsilon)}{\partial y}}$$

where $Y^1 = Y^1(x, y, \underset{1}{y}; \varepsilon)$ *is defined by 13 ,* $Y^{k-1} = Y^{k-1}(x, y, \underset{1}{y}, \cdots, \underset{k-1}{y}; \varepsilon)$ *and* $\underset{q}{y} = \frac{d^q y}{dx^q}$ *and* $\underset{q}{y}^* = \frac{d^q y^*}{dx^{*q}}$*,* $q =$ *1,2,3,....*

For the complete proof follow [21] page 56. Using theorem 3.2 we can define the k^{th} extended infinitesimal generator of 12 and we can use this extended infinitesimal generator to find infinitesimals.
By expanding k^{th} extension of 12 in Taylor series about $\varepsilon = 0$, we can obtain .

$$x^* = X(x, y; \varepsilon) = x + \varepsilon.\xi(x, y) + O(\varepsilon^2)$$

$$y^* = Y(x, y; \varepsilon) = y + \varepsilon.\eta(x, y) + O(\varepsilon^2)$$

$$\underset{1}{y}^* = Y^1(x, y, \underset{1}{y}; \varepsilon) = \underset{1}{y} + \varepsilon\eta^1(x, y, \underset{1}{y}) + O(\varepsilon^2)$$

$$\vdots$$

$$\underset{k}{y}^* = Y^k(x, y, \underset{1}{y}, ..., \underset{k}{y}; \varepsilon) = \underset{k}{y} + \varepsilon\eta^k(x, y, \underset{1}{y}, \underset{2}{y}, ..., \underset{k}{y}) + O(\varepsilon^2) \tag{14}$$

Therefore the infinitesimals of extended Lie group of transformation is given by

$$(\xi(x, y), \eta(x, y), \eta^1(x, y, \underset{1}{y}), ..., \eta^k(x, y, \underset{1}{y}, \underset{2}{y}, ..., \underset{k}{y}))$$

and the k^{th} extended infinitesimal generator is given by

$$\bar{X}^k = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} + \eta^1(x, y, \underset{1}{y})\frac{\partial}{\partial \underset{1}{y}} + ... + \eta^k(x, y, \underset{1}{y}...\underset{k}{y})\frac{\partial}{\partial \underset{k}{y}} \tag{15}$$

Theorem 3.3. *Let* $\eta^k(x, y, \underset{1}{y}, \underset{2}{y}, ..., \underset{k}{y})$ *be the infinitesimal provided by the transformation* $\underset{1}{y}^* = Y^k(x, y, y^1, ..., y^k; \varepsilon)$ *defined in (11). Then*

$$\eta^k(x, y, \underset{1}{y}, \underset{2}{y}, ..., \underset{k}{y}) = D_x.\eta^{k-1} - \underset{k}{y}.D_x\xi \tag{16}$$

where D_x *is total derivative operator with respect to* x

Above theorem has been interpreted from [20]. This theorem is used to obtain explicit formulas for the infinitesimals of extended transformations.

Theorem 3.4. *Consider the Lie group of transformations 12 admitted by* k^{th} $(k > 1)$ *order ODE. Let* $(r(x, y), s(x, y))$ *be corresponding canonical coordinates satisfying* $\boldsymbol{X}r = 0$ *and* $\boldsymbol{X}s = 1$*. Then solving the* k^{th} *order ODE reduced to solving* $(k-1)^{th}$ *order ODE.*

$$\frac{d^{k-1}u}{dr^{k-1}} = G(r, u, \frac{du}{dr}, ..., \frac{d^{k-2}u}{dr^{k-2}})$$

where $\frac{ds}{dr} = u$ *and* $\bar{X}$ *is the infinitesimal generator.*

For the complete proof of theorem 3.4 follow [21] page 111. According to this theorem we can reduce the order of a given higher order NODE using the canonical coordinates of a Lie symmetry group of the NODE.

Perera & Gallage, *Adv. J. Grad. Res.*; Vol. 13, Issue 1, pp: 37-61, January 2023

4 Linearized Symmetry Condition of ODEs

In this chapter for first order ODEs and higher order ODEs two different equations are presented to obtain general solutions for infinitesimals of their Lie symmetry groups.

4.1 Linearized Symmetry Condition of first order ODEs

This session contains the construction of linearized symmetry condition of higher order ODEs.

Consider the 1^{st} order ODE given by $\frac{dy}{dx} = g(x,y)$. Let 1 be a symmetry group of $f(x,y)$. According to the symmetry condition 2

$\frac{dy^*}{dx^*} = g(x^*,y^*)$ when $\frac{dy}{dx} = g(x,y)$

Then by total derivative operator

$$\frac{D_y y^*}{D_x x^*} = \frac{y^*_x + y' y^*_y}{x^*_x + y' x^*_y} = g(x^*,y^*) \tag{17}$$

By expanding x^*, y^* and $g(x^*,y^*)$ in Taylor series about $\varepsilon = 0$, and ignoring terms of order ε^2 and higher

$$\begin{aligned} x^* &= x + \varepsilon\xi(x,y) \\ y^* &= y + \varepsilon\eta(x,y) \\ g(x^*,y^*) &= g(x,y) + \varepsilon(g_x(x,y)\xi(x,y) + g_y(x,y)\eta(x,y)) \end{aligned} \tag{18}$$

Then by substituting equations 18 into 17

$$\eta_x - \xi_y g^2 + (\eta_y - \xi_x)g = \xi g_x + \eta g_y \tag{19}$$

Linearized symmetry condition of first order differential equations is given by equation 19. Above construction of equation 19 is represented from the content of [20] and [27].

Example 4.1. Consider the non linear ordinary differential equation $\frac{dy}{dx} = g(x,y) = 2y + xy^2$ [17] Then the Linearized symmetry condition 19 becomes

$$\eta_x - \xi_y(2y + xy^2)^2 + (\eta_y - \xi_x)(2y + xy^2) = \xi(y^2) + \eta(2 + 2xy)$$

Appropriate assumptions for the forms of the infinitesimals should be made to solve this symmetry condition otherwise it is impossible to solve the equation [27]. Assume that $\xi(x,y) = I(x)$ and $\eta(x,y) = J(x)y^2$ where I and J are functions of x. Then Linerized symmetry condition becomes

$$J'y^2 + (2Jy - I')(2y + xy^2) - I(y^2) - (J)(y^2)(2 + 2xy) = 0 \tag{20}$$

Then by comparing coefficients of y in R.H.S and L.H.S of 20 we can obtain

$$2I' = 0$$

Hence

$$I = c, \qquad c \in \mathbb{R}$$

Then by substituting $I = c, c \in \mathbb{R}$ to 20 and comparing coefficients of y in R.H.S and L.H.S of 20 we can obtain

$$J' + 2J - c = 0$$

Hence

$$J = \frac{c}{2}$$

Therefore $\xi(x,y) = I(x) = c$ and $\eta(x,y) = J(x)y^2 = \frac{cy^2}{2}$

5 Linerized Symmetry Condition of Second and Higher Order ODEs

This session contains the construction of linearized symmetry condition of higher order ODEs. Following theorem is interpreted based on [21]

Theorem 5.1. *Consider the one parameter Lie group of transformations 1with the infinitesimal generator* $\boldsymbol{X} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$. *Let* $\boldsymbol{X}^k = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \eta^1(x,y,\underset{1}{y})\frac{\partial}{\partial \underset{1}{y}} + ... + \eta^k(x,y,\underset{1}{y},...,\underset{k}{y})\frac{\partial}{\partial \underset{k}{y}}$ *be the* k^{th} *extended infinitesimal generator of 1. Then the* k^{th} *order ODE* $\underset{k}{y} = f(x,y,\underset{1}{y},\underset{2}{y},...,\underset{k-1}{y})$ *admits this one parameter Lie group of transformations if and only if* $\boldsymbol{X}^k(\underset{k}{y} - f(x,y,\underset{1}{y},...,\underset{k-1}{y})) = 0$

Then by expanding $\mathbf{X}^k(\underset{k}{y} - f(x,y,\underset{1}{y},...,\underset{k-1}{y})) = 0$,we can obtain

$$\eta^k - [\xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \eta^1\frac{\partial f}{\partial \underset{1}{y}} + ... + \eta^{k-1}\frac{\partial f}{\partial \underset{k-1}{y}}] = 0 \qquad (21)$$

The equation 21 is called Linearized symmetry condition of second and higher order ordinary ODEs. For a given NODE we can use this Linearized symmetry condition to find general solutions for the infinitesimals of its Lie symmetry groups.

6 Applications

In this session we present examples of solving first and second order NODEs using Lie symmetry groups. In this session we use following notations. $\underset{q}{y} = \frac{d^q y}{dx^q}$, q = 1,2,3,.... and $\underset{1}{y} = y' = \frac{dy}{dx}$.

6.1 First Order NODEs.

This section contains the examples of finding exact solutions for the first order NODEs using Lie symmetry groups.

6.1.1 Method 1 [Using Canonical Coordinates]

In the first method for solving first order NODEs, we use canonical coordinates to convert a given NODEs into variable separable form [17],[20],[27].

Example 6.1. In this example , we obtain exact solutions for a NODE using canonical coordinates.
Consider the ODE [17]

$$\frac{dy}{dx} = 2y + xy^2$$

In example 4.1 we obtained general solutions for the infinitesimals of its Lie symmetry groups. $\xi(x,y) = c$ and $\eta(x,y) = \frac{cy^2}{2}$. Let $c = 1$.Then the infinitesimals become $(1, \frac{y^2}{2})$ .Then the infinitesimal generator is given by

$$\bar{X} = \frac{\partial}{\partial x} + \frac{y^2}{2}\frac{\partial}{\partial y}$$

According to theorem 2.11 there exists canonical coordinates (r(x,y),s(x,y)) such that the Lie symmetry group becomes

$$r^* = r \qquad s^* = s + \varepsilon$$

where ε is a real parameter. Then the equations 8 and 9 become

$$\frac{\partial r}{\partial x} + \frac{y^2}{2}\frac{\partial r}{\partial y} = 0$$

$$\frac{\partial s}{\partial x} + \frac{y^2}{2}\frac{\partial s}{\partial y} = 1$$

The characteristic equation $\frac{dx}{1} = 2\frac{dy}{y^2}$ corresponds to the partial differential equation $\frac{\partial r}{\partial x} + \frac{y^2}{2}\frac{\partial r}{\partial y} = 0$. Hence we find r by solving the characteristic equation $\frac{dx}{1} = 2\frac{dy}{y^2}$.

$$\int \frac{dx}{1} = \int \frac{2dy}{y^2}$$
$$x = \frac{-2}{y} + c, c \in \mathbf{R}$$
$$c = \frac{xy+2}{y}, \quad c \in \mathbb{R}$$

Therefore, according to method of characteristics $r(x,y) = c = \frac{xy+2}{2}$. Similarly, we find the dependent coordinate s by solving the characteristic equation $\frac{dx}{1} = \frac{2dy}{y^2} = \frac{ds}{1}$.

$$s(x,y) = \int \frac{ds}{1} = \int \frac{dx}{1}$$
$$s(x,y) = x + c \qquad ,c \in \mathbb{R}$$

Hence the canonical coordinates are given by $(r(x,y), s(x,y) = (\frac{xy+2}{y}, x)$. Then we can convert the NODE into variable separable form using these canonical coordinates. By total derivative operator

$$\frac{ds}{dr} = \frac{s_x + y's_y}{r_x + y'r_y}$$
$$= \frac{1}{1 + (2y - xy^2)\frac{-2}{y^2}}$$
$$= \frac{1}{(1-2r)}$$

Therefore the given NODE exists in a variable separable form in trems of canonical coordinates. By integration.

$$\int ds = \int \frac{1}{(1-2r)}$$
$$s = \frac{-1}{2} ln(2r-1) + c1 \quad ,c1 \in \mathbb{R}$$

We obtain exact solution of the given NODE by converting the canonical coordinates to original coordinates.

$$y = \frac{4}{1 + e^{-2x}c2 - 2x}, c2 \in \mathbb{R}$$

The invariant solution curves for the corresponding Lie symmetry group can be obtained by solving 10 for y.

$$\eta(x,y) - \xi(x,y)y' = 0$$
$$\frac{y^2}{2} - (2y - xy^2) = 0$$

Then the invariant solution curves are given by

$$y_1 = 0 \qquad y_2 = \frac{-4}{2x-1}$$

Figure 3 illustrates how theses invariant solution curves exits in vector field $(1, \frac{y^2}{2})$ in a way that no flow curve cross them. In Figure 3 red curves represents the invariant solution curve y_1, blue curves represents the invariant solution curve y_2 and the black curves represent flow curves.

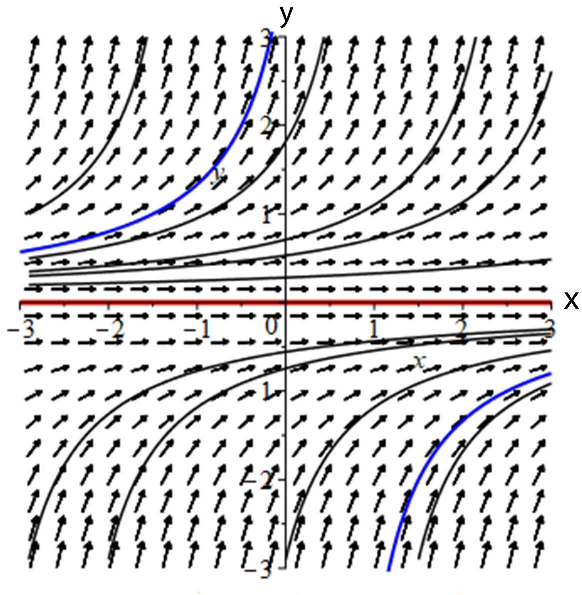

Figure 3: vector field $(1, \frac{y^2}{2})$ with flow curves and curves $y = 0$, $y = \frac{-4}{2x-1}$.

6.1.2 Method 2 [Using Invariant Curves]

The concept of applying invariant curves to get general solutions is given in [20] [19]. In this session we link definitions and theorems including fundamental theorem of Lie,taken from different references to present all the steps of applying this concept to NODEs as a complete method. In this method first we obtain two different sets of infinitesimals for the given NODE by solving symmetry condition 19 . Then one set of infinitesimals are applied to 10 to obtain invariant curves and remaining set is applied to first fundamental theorem of Lie to get the corresponding symmetry group. Then we obtain the exact solution by applying the symmetry group to the invariant curve.

Example 6.2. For this example a differential equation is taken from the exercises of [17] to obtain general solution using invariant solution curves.
Consider the NODE.

$$\frac{dy}{dx} = \frac{y}{x} + \frac{x^2}{x+y}$$

It has two different Lie symmetry groups with different tangent vector fields $(1, \frac{y}{x})$ and $(0, \frac{-(2x^3-2xy-y^2)}{x+y})$. These different sets of infinitesimals can be obtained by solving linearized symmetry condition with two different assumptions for the form of the infinitesimals or using Maple [28, 29].
We can find the Lie symmetry group whose tangent vector filed is $(1, \frac{y}{x}) = (\xi(x, y), \eta(x, y))$ using the First fundamental theorem of Lie.

Let $v^* = \psi(v, \varepsilon)$ be a Lie symmetry group whose tangent vector field is $(1, \frac{y}{x})$ where $v^* = (x^*, y^*), v = (x, y)$ and ε is a real parameter. By 6

$$\frac{dx^*}{d\varepsilon} = \xi(x^*, y^*) \qquad \frac{dy^*}{d\varepsilon} = \eta(x^*, y^*)$$

We can obtain x^* and y^* by solving above two ODEs

$$\frac{dx^*}{d\varepsilon} = \xi(x^*, y^*) = 1$$
$$\int dx^* = \int d\varepsilon$$
$$x^* + c(x, y) = \varepsilon$$

Perera & Gallage, *Adv. J. Grad. Res.*; Vol. 13, Issue 1, pp: 37-61, January 2023

$v^* = v$ when $\varepsilon = 0$.Hence $x^* = x$ when $\varepsilon = 0$.
Then

$$x + c(x,y) = 0$$
$$c(x,y) = -x$$

Therefore

$$x^* - x = \varepsilon$$
$$x^* = x + \varepsilon$$

Similarly we can find y^*

$$\frac{dy^*}{d\varepsilon} = \eta(x^*, y^*) = \frac{y^*}{x^*}$$
$$\int \frac{1}{y^*} dy^* = \int \frac{1}{x+\varepsilon} d\varepsilon$$
$$ln(y^*) + c(x,y) = ln(x+\varepsilon)$$

$v^* = v$ when $\varepsilon = 0$.Hence $y^* = y$ when $\varepsilon = 0$. Using this initial condition we can obtain

$$c(x,y) = ln(\frac{x}{y})$$

Hence

$$ln(y^*) + ln(\frac{x}{y}) = ln(x+\varepsilon)$$
$$y^* = \frac{(x+\varepsilon)y}{x}$$

Next we find the invariant curves of the given NODE under the Lie symmetry group whose tangent vector field is $(0, \frac{-(2x^3-2xy-y^2)}{x+y})$ using 10.

$$\frac{-(2x^3 - 2xy - y^2)}{x+y} - 0(\frac{y}{x} + \frac{x^2}{x+y}) = 0$$
$$\frac{-2x^3 + 2xy + y^2}{x+y} = 0$$

Then the invariant solution curves are given by

$$y_1 = -x + (\sqrt{1+2x})x \qquad y_2 = -x - (\sqrt{1+2x})x$$

Under transformed variables x^*, y^* solution curves become

$$y^* = -x^* + (\sqrt{1+2x^*})x^* \qquad y^* = -x^* - (\sqrt{1+2x^*})x^*$$

Consider the solution curve $y^* = -x^* + (\sqrt{1+2x^*})x^*$.

$$\frac{(x+\varepsilon)y}{x} = -(x+\varepsilon) + (\sqrt{1+2(x+\varepsilon)})(x+\varepsilon)$$
$$y = (-1 + \sqrt{1+2\varepsilon+2x})x$$

Similarly we can obtain $y = (-1 - \sqrt{1+2\varepsilon+2x})x$ using the invariant curve $y = -x - (\sqrt{1+2x})x$.
Let $c = \varepsilon$.Then the a general solution is given by

$$y = (-1 \pm \sqrt{1+2c+2x})x \qquad , c \in \mathbb{R}$$

Example 6.3. For this example a differential equation is taken from the exercises of [17] to obtain general solution using invariant solution curves.
Consider the NODE

$$\frac{dy}{dx} = \frac{1-y-2xy^2}{x(2xy+1)}$$

It has two different Lie symmetry groups with different tangent vector fields $(1, -\frac{y}{x})$ and $(0, \frac{xy^2+y-1}{2xy+1})$. These different sets of infinitesimals can be obtained by solving linearized symmetry condition with two different assumptions for the form of the infinitesimals or using Maple [28, 29]. We can find the Lie symmetry group whose tangent vector filed is $(1, -\frac{y}{x}) = (\xi(x,y), \eta(x,y))$ using the first fundamental theorem of Lie.

Let $v^* = \psi(v, \varepsilon)$ be a Lie symmetry group of the NODE whose tangent vector field is $(1, \frac{y}{x})$ where $v^* = (x^*, y^*), v = (x, y)$ and ε is a real parameter. By 6

$$\frac{dx^*}{d\varepsilon} = \xi(x^*, y^*) \qquad \frac{dy^*}{d\varepsilon} = \eta(x^*, y^*)$$

Therefore we can obtain x^* and y^* by solving above two ODEs.

$$\begin{aligned} \frac{dx^*}{d\varepsilon} &= \xi(x^*, y^*) = 1 \\ \int dx^* &= \int d\varepsilon \\ x^* + c(x,y) &= \varepsilon \end{aligned}$$

$x^* = x$ when $\varepsilon = 0$. Then

$$\begin{aligned} x + c(x,y) &= 0 \\ c(x,y) &= -x \end{aligned}$$

Hence

$$\begin{aligned} x^* - x &= \varepsilon \\ x^* &= x + \varepsilon \end{aligned}$$

Similarly we can find y^*

$$\begin{aligned} \frac{dy^*}{d\varepsilon} &= \eta(x^*, y^*) = -\frac{y^*}{x^*} \\ -\int \frac{1}{y^*} dy^* &= \int \frac{1}{x+\varepsilon} d\varepsilon \\ -ln(y^*) + c(x,y) &= ln(x+\varepsilon) \end{aligned}$$

$y^* = y$ when $\varepsilon = 0$.Using this initial value we can obtain

$$c(x,y) = ln(xy)$$

Hence

$$\begin{aligned} -ln(y^*) + ln(xy) &= ln(x+\varepsilon) \\ y^* &= \frac{xy}{(x+\varepsilon)} \end{aligned}$$

Then we can find the invariant curves of the given NODE under the Lie symmetry group whose tangent vector field is $(0, \frac{xy^2+y-1}{2xy+1})$ using theorem 2.12.

$$\begin{aligned} \frac{xy^2+y-1}{2xy+1} - 0(\frac{y}{x} + \frac{x^2}{x+y}) &= 0 \\ \frac{xy^2+y-1}{2xy+1} &= 0 \end{aligned}$$

Then the invariant solution curves are given by

$$y = \frac{-1+\sqrt{1+4x}}{2x} \qquad y = \frac{-1-\sqrt{1+4x}}{2x}$$

Under transformed variables x^*, y^* solution curves become

$$y^* = \frac{-1+\sqrt{1+4x^*}}{2x^*} \qquad y^* = \frac{-1-\sqrt{1+4x^*}}{2x^*}$$

Consider the solution curve $y^* = \frac{-1+\sqrt{1+4x^*}}{2x^*}$.

$$\frac{xy}{x+\varepsilon} = \frac{-1+\sqrt{1+4(x+\varepsilon)}}{2(x+\varepsilon)}$$
$$y = \frac{-1+\sqrt{1+4\varepsilon+4x}}{2x}$$

Similarly we can obtain $y = \frac{-1-\sqrt{1+4\varepsilon+4x}}{2x}$ using the invariant curve $y = \frac{-1-\sqrt{1+4x}}{2x}$.
Let $c = \varepsilon$. Then the a general solution is given by

$$y = \frac{-1 \pm \sqrt{1+4c+4x}}{2x} \qquad , c \in \mathbb{R}$$

6.2 Second and Higher Order NODEs

In this session we use Lie symmetry groups of higher order NODEs to reduce the order [17], [21] and we use one of two methods presented above to solve the reduced NODE.

The linearized symmetry condition 21 is used to find the infinitesimals of Lie symmetry group that admitted by the given NODE. Explicit formulas for the infinitesimals η^k (k= 1,2,3...) defined in 14 are required to apply for the equation 21. Therefore we can use theorem 3.3 to obtain explicit formulas for η^k.

Since in this thesis Lie symmetry groups are applied only up to second order NODEs we find explicit formals only for η^1 and η^2.

By theorem 3.3

$$\begin{aligned}\eta^1 &= \frac{D\eta}{Dx} - \underset{1}{y}\frac{D\xi(x,y)}{Dx} \\ &= \frac{\partial\eta}{\partial x} + \underset{1}{y}\frac{\partial\eta}{\partial y} - \underset{1}{y}(\frac{\partial\xi}{\partial x} + \underset{1}{y}\frac{\partial\xi}{\partial y}) \\ &= \eta_x + \underset{1}{y}\eta_y - \underset{1}{y}(\xi_x + \underset{1}{y}\xi_y)\end{aligned}$$

$$\eta^1 = \eta_x + (\eta_y - \xi_x)\underset{1}{y} - \xi_y(\underset{1}{y})^2 \tag{22}$$

Similarly we can obtain the explicit formula for η^2

$$\eta^2 = \eta_{xx} + (2\eta_{xy} - \xi_{xx})\underset{1}{y} + (\eta_{yy} - 2\xi_{xy})(\underset{1}{y})^2 - \xi_{yy}(\underset{1}{y})^3 + (\eta_y - 2\xi_x)\underset{2}{y} - 3\xi_y\underset{1}{y}\underset{2}{y} \tag{23}$$

Example 6.4. For this example a second order NODE is taken from the exercises of [17]. Consider the second order NODE

$$\frac{d^2y}{dx} = \frac{-2x}{y^2}(\frac{dy}{dx})^3 = f(x,y,\underset{1}{y})$$

The linearized symmetry condition 21 becomes

$$\eta^2(x,y,\underset{1}{y},\underset{2}{y}) - [\xi(x,y)\frac{\partial f}{\partial x} + \eta(x,y)\frac{\partial f}{\partial y} + \eta^1(x,y,\underset{1}{y})\frac{\partial f}{\partial \underset{1}{y}}] = 0$$

By substituting formulas 22 and 23 we can obtain
$\eta_{xx} + (2\eta_{xy} - \xi_{xx})\underset{1}{y} + (\eta_{yy} - 2\xi_{xy})(\underset{1}{y})^2 - \xi_{yy}(\underset{1}{y})^3 + (\eta_y - 2\xi_x - 3\xi_y \underset{1}{y})\underset{2}{y} - [\xi(x,y)\frac{\partial f}{\partial x} + \eta(x,y)\frac{\partial f}{\partial y} + (\eta_x + (\eta_y - \xi_x)\underset{1}{y} - \xi_y(\underset{1}{y})^2)\frac{\partial f}{\partial \underset{1}{y}}] = 0$

By substituting $\underset{2}{y} = f = \frac{-2x}{y^2}(y')^3$,$\frac{\partial f}{\partial y} = \frac{4x}{y^3}(y')^3$, $\frac{\partial f}{\partial x} = \frac{-2(y')^3}{y^2}$, $\frac{\partial f}{\partial \underset{1}{y}} = \frac{-6x(y')^2}{y^2}$ and $\underset{1}{y} = y'$ we can obtain.

$\eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})(y')^2 - \xi_{yy}(y')^3 + (\eta_y - 2\xi_x - 3\xi_y y')(\frac{-2x}{y^2}(y')^3) - [\xi(x,y)(\frac{-2(y')^3}{y^2}) + \eta(x,y)(\frac{4x}{y^3}(y')^3) + (\eta_x + (\eta_y - \xi_x)y' - \xi_y(y')^2)(\frac{-6x(y')^2}{y^2})] = 0$
Then by comparing powers of y' we can obtain following equations called determining equations.

$$
\begin{aligned}
-(\xi_{yy})y^3 - 2\xi_x xy + 4\eta_y xy + 2\xi y - 4x\eta &= 0 \\
(\eta_{yy})y^3 - 2(\xi_{xy})y^3 + 6\eta_x xy &= 0 \\
2(\eta_{xy})y^3 - (\xi_{xx})y^3 &= 0 \\
(\eta_{xx})y^3 &= 0
\end{aligned}
$$

By solving these set of determining equations we can obtain general expressions for $\xi(x,y)$ and $\eta(x,y)$. we can use Maple to solve these set of determining equations for $\xi(x,y)$ and $\eta(x,y)$ [10] [30] [31] [7]

$$
\begin{aligned}
\xi(x,y) &= 2x^2yc_1 - \frac{x^2c_2}{y^2} + (2y^3c_4 - \frac{c_5}{y^3} + c_6)x + c_7y^2 + \frac{c_8}{y} \\
\eta(x,y) &= \frac{c_4y^6 + c_1xy^4 + c_3y^3 + c_2xy + c_5}{y^2}
\end{aligned}
$$

Where $c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8 \in \mathbb{R}$. Then by changing the constants $c_1, c_2, c_3, c_4, c_5, c_6, c_7$ and c_8 we can obtain different sets of infinitesimals for the given ODE. Let $c_1 = 0, c_2 = 0, c_3 = 0, c_4 = 0, c_5 = 0, c_6 = 0, c_8 = 0$ and $c_7 = 1$. Then the infinitesimals becomes

$$\xi(x,y) = y^2 \qquad \eta(x,y) = 0$$

Then we can conclude that the given ODE has a Lie symmetry group whose tangent vector field is $(y^2, 0)$. Then the infinitesimal generator is given by

$$\mathbf{X} = y^2\frac{\partial}{\partial x}$$

By theorem 2.11 there exists canonical coordinates (r(x,y),s(x,y)) such that the corresponding Lie symmetry group becomes

$$r^* = r \qquad s^* = s + \varepsilon$$

Where ε is areal parameter. Then the equations 8 and 9 become

$$y^2\frac{\partial r}{\partial x} = 0$$

$$y^2\frac{\partial s}{\partial x} = 1$$

Then we can obtain the canonical coordinates $(r(x,y), s(x,y)) = (y, \frac{x}{y^2})$ of corresponding Lie symmetry group by solving above two equations.

By theorem 3.4 we can convert the given second order ordinary differential equation into first order ordinary differential equation by changing the coordinates x, y into canonical coordinates and making $\frac{ds}{dr} = u$.

Perera & Gallage, *Adv. J. Grad. Res.*; Vol. 13, Issue 1, pp: 37-61, January 2023

Reduced ODE can be obtain by converting each term of $\frac{d^2y}{dx} = \frac{-2x}{y^2}(\frac{dy}{dx})^3$ into canonical coordinates.

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dr}/\frac{dx}{dr}\\ &= \frac{1}{2rs + r^2s'}\\ \frac{d^2y}{dx^2} &= \frac{dy'}{dr}/\frac{dx}{dr}\\ &= \frac{-(2s + 2rs' + 2rs' + r^2s'')}{(2rs + r^2s')^3}\end{aligned}$$

Therefore the ODE $\frac{d^2y}{dx} = \frac{-2x}{y^2}(\frac{dy}{dx})^3$ becomes

$$\begin{aligned}\frac{-(2s + 2rs' + 2rs' + r^2s'')}{(2rs + r^2s')^3} &= \frac{-2r^2s}{r^2(2rs + r^2s')^3}\\ 2s + 2rs' + 2rs' + r^2s'' &= 2s\\ s'' &= \frac{-4s'}{r}\end{aligned}$$

Then we can reduce the order using $s' = u$

$$u' = \frac{-4u}{r}$$

Then by integration we can obtain the solution

$$u = \frac{c}{r^4} \quad , c \in \mathbb{R}$$

Since $u = s'$

$$\frac{ds}{dr} = \frac{c}{r^4}$$

Then by integration we can obtain the solution

$$s = \frac{-c}{3r^3} + c_9 \quad , c_9 \in \mathbb{R}$$

By changing canonical coordinates to original coordinates (x, y) we can obtain the general solution

$$\frac{x}{y^2} = \frac{-c}{3(y^3)} + c_9$$

Example 6.5. For this example a second order NODE is taken from the exercises of [20]. Consider the following second order NODE.

$$\frac{d^2y}{dx^2} = \frac{3}{2y}(\frac{dy}{dx})^2 + 2y^3 = f(x, y, \underset{1}{y})$$

The linearized symmetry condition 21 becomes

$$\eta^2(x, \underset{1}{y}, \underset{2}{y}) - [\xi(x,y)\frac{\partial f}{\partial x} + \eta(x,y)\frac{\partial f}{\partial y} + \eta^1(x, y, \underset{1}{y})\frac{\partial f}{\partial \underset{1}{y}}] = 0$$

By substituting formulas formulas 22 and 23 we can obtain
$\eta_{xx} + (2\eta_{xy} - \xi_{xx})\underset{1}{y} + (\eta_{yy} - 2\xi_{xy})(\underset{1}{y})^2 - \xi_{yy}(\underset{1}{y})^3 + (\eta_y - 2\xi_x - 3\xi_y\underset{1}{y})\underset{2}{y} - [\xi(x,y)\frac{\partial f}{\partial x} + \eta(x,y)\frac{\partial f}{\partial y} + (\eta_x + (\eta_y - \xi_x)\underset{1}{y} - \xi_y(\underset{1}{y})^2)\frac{\partial f}{\partial \underset{1}{y}}] = 0$

By substituting $\underset{2}{y} = f = \frac{3}{2y}(y')^2 + 2y^3, \frac{\partial f}{\partial y} = \frac{-3(y')^2}{2y^2} + 6y^2$, $\frac{\partial f}{\partial \underset{1}{y}} = \frac{-3y'}{y}$ and $\underset{1}{y} = y'$ we can obtain

$\eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})(y')^2 - \xi_{yy}(y')^3 + (\eta_y - 2\xi_x - 3\xi_y y')(\frac{3}{2y}(y')^2 + 2y^3) - [\eta(x,y)(\frac{-3y'^2}{2y^2} + 6y^2) + (\eta_x + (\eta_y - \xi_x)y' - \xi_y(y')^2)(\frac{-3y'}{y})] = 0$

Then by comparing powers of y' we can obtain following equations called determining equations.

$$-2\xi_{yy}y^2 - 3\xi_y y = 0$$
$$-4\xi_{x,y}y^2 + 2\eta_{yy}y^2 - 3\eta_y y + 3\eta = 0$$
$$-12\xi_y y^5 + 4\eta_{xy}y^2 - 2\xi_{xx}y^2 - 6\eta_x y = 0$$
$$-8\eta_x y^5 + 4\eta_x y^5 - 12\eta y^4 + 2\eta_{xx}y^2 = 0$$

By solving these set of determining equations we can obtain general expressions for $\xi(x,y)$ and $\eta(x,y)$. we can use Maple to solve these set of determining equations for $\xi(x,y)$ and $\eta(x,y)$ [10] [30] [31] [7]

$$\xi(x,y) = \frac{1}{2}c_1x^2 + c_2x + c_3$$
$$\eta(x,y) = -(c_1x + c_2)y$$

where $c_1, c_2, c_3 \in \mathbb{R}$.

Then by changing the constants c_1, c_2 and c_3 we can obtain different sets of infinitesimals for the given ODE. Let $c_1 = 0, c_2 = 0$ and $c_3 = 1$. Then the infinitesimals become

$$\xi(x,y) = 1 \qquad \eta(x,y) = 0$$

Then we can conclude that the given ODE has a Lie symmetry group whose tangent vector field is $(1,0)$. Then the infinitesimal generator is given by

$$\mathbf{X} = 1\frac{\partial}{\partial x}$$

By theorem 2.11 there exists canonical coordinates (r(x,y),s(x,y)) such that the corresponding Lie symmetry group becomes

$$r^* = r \qquad s^* = s + \varepsilon$$

ε is areal parameter. Then the equations 8 and 9 become

$$1\frac{\partial r}{\partial x} = 0$$

$$1\frac{\partial s}{\partial x} = 1$$

Then we can obtain the canonical coordinates of corresponding Lie symmetry group by solving above two equations . $(r(x,y), s(x,y)) = (y,x)$

By theorem 3.4 we can convert the given second order ordinary differential equation into firs order ordinary differential equation by changing the coordinates x, y into canonical coordinates and making $\frac{ds}{dr} = u$.

Reduced ODE can be obtain by converting each term of $\frac{d^2y}{dx^2} = \frac{3}{2y}\frac{dy}{dx} + 2y^3$ by canonical coordinates.

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dr}/\frac{dx}{dr}\\ &= \frac{1}{s'}\\ \frac{d^2y}{dx^2} &= \frac{dy'}{dr}/\frac{dx}{dr}\\ &= \frac{-s''}{s'^3}\end{aligned}$$

Therefore the ODE $\frac{d^2y}{dx^2} = \frac{3}{2y}\frac{dy}{dx} + y^3$ becomes

$$\frac{-s''}{(s')^3} = \frac{3}{2r(s')^2} + 2r^3$$
$$s'' = -\frac{3s'}{2r} - (s')^3r^3$$

Then we can reduce the order using $s' = u$

$$u' = -\frac{3u}{2r} - 2u^3r^3$$

To get the exact solution we can solve the first order ODE $u' = -\frac{3u}{2r} - 2u^3r^3$ using Lie symmetry method. Consider the first order ordinary differential equation $u' = -\frac{3u}{2r} - 2u^3r^3$. It has a Lie symmetry group whose tangent vector field is $(\xi(r,u), \eta(r,u)) = (r, -2u)$. These infinitesimals can be obtained by solving the linearized symmetry condition of u' or using maple [28, 29].Then the infinitesimal generator is given by

$$\mathbf{X} = r\frac{\partial}{\partial r} - 2u\frac{\partial}{\partial u}$$

By theorem 2.11 there exists canonical coordinates (p(x,y),q(x,y)) such that the Lie symmetry group becomes

$$p^* = p$$
$$q^* = q + \varepsilon$$

Whereε is a real parameter.
Then the equations 8 and 9 become

$$r\frac{\partial p}{\partial r} - 2u\frac{\partial p}{\partial u} = 0$$
$$r\frac{\partial q}{\partial r} - 2u\frac{\partial q}{\partial u} = 1$$

Then we can find independent canonical coordinate p by solving the characteristic equation

$$\frac{dr}{r} = \frac{du}{-2u}$$
$$\int \frac{1}{r}dr = \int \frac{1}{-2u}du$$
$$ln(r^{-2}) + ln(c) = ln(u) \quad ,c \in \mathbb{R}$$
$$c = r^2u$$

Then $p(r,u) = c = r^2u$. we can find the dependent canonical coordinate q , by solving the characteristic equation .

$$\frac{dr}{r} = \frac{du}{-2u} = dq$$
$$q = \int dq = \int \frac{dr}{r}$$
$$q = ln(r) + c_1 \quad c_1 \in \mathbb{R}$$

Hence the canonical coordinates are given by $(p(r,u), q(r,u)) = (r^2u, ln(r))$. Then we can convert the ODE into variable separable form using these canonical coordinates. By total derivative operator

$$\frac{dq}{dp} = \frac{q_r + u'q_u}{p_r + u'p_u}$$
$$= \frac{\frac{1}{r}}{2ru + (-\frac{3u}{2r} - 2r^3u^3)r^2}$$
$$= \frac{2}{r^2u - 4(r^2u)^3}$$
$$= \frac{2}{p - 4p^3}$$

By integration we can obtain the solution

$$q = -ln(2p-1) - ln(2p+1) + 2ln(p) + c \qquad , c \in \mathbb{R}$$

By changing canonical coordinates to coordinates (r, u) we can obtain the general solutions.

$$u = \pm \frac{1}{\sqrt{c_1 r + 4r^2 r}} \qquad , c_1 \in \mathbb{R}$$

Consider the general solution $u = \frac{1}{\sqrt{c_1 r + 4r^2 r}}$. Since $u = \frac{ds}{dr}$

$$\frac{ds}{dr} = \frac{1}{\sqrt{c_1 r + 4r^2 r}}$$

Then by integration we can obtain

$$s = -\frac{2(4x + c_1)}{c_1 \sqrt{x(4x + c_1)}} + c_2 \qquad , c_2 \in \mathbb{R}$$

Then by changing canonical coordinates (r, s) to original coordinates (x, y), we can obtain the general solution to the Given second order non linear differential equation .

$$y = \frac{4c_1}{c_1^2 c_2^2 - 2c_1^2 c_2 x + c_1^2 x^2 - 16}$$

Similarly we can obtain this general solution using $u = -\frac{1}{\sqrt{c_1 r + 4r^2 r}}$.

According to theorem 3.4 we are able to reduce the order of higher order NODEs in the same way that we used in examples 6.5 and 6.4.

7 Discussion

The applications of Lie symmetry analysis have been used in many research for solving nonlinear partial differential equations found in physical problems [2, 26, 4, 7, 3, 6, 32, 8] and for mathematical Models such as SIR models [33]. But in this project, we put a significant emphasis on discussing the idea of symmetries of ODEs and demonstrate how their features can be used to obtain the exact solutions. For that, we have discussed the relation between vector fields and the symmetries of ODEs and we have illustrated symmetries and their vector fields graphically in Figure 1, Figure 2 and Figure 3. Moreover, we have explained the theorems and definitions of Lie symmetry analysis from different references and combined them to present, three different complete solution methods.

In example 6.1 we have obtain general solutions for NODEs by converting them into variable separable form using canonical coordinates of their Lie symmetry groups. Although we are able to apply this method to any first order NODE, we have to find appropriate assumption to the form of the infinitesimals to solve the symmetry condition. For some NODEs finding these correct assumptions can be very hard and complex.

In examples 6.2 and 6.3 we have solved NODEs using invariant solution curves under Lie symmetry groups. In this method, we have to find two different sets of assumptions for the infinitesimals, since we use two different sets of infinitesimals. Furthermore, invariant curves that we obtain using one set of infinitesimals can't be invariant curves under the OLGT that we obtain by applying the remaining set of infinitesimals to the first fundamental theorem of Lie. Moreover, we have to find appropriate Lie symmetry groups which give real invariant solutions, because for some NODEs most of its Lie symmetry groups give complex invariant solutions. Therefore we have to find assumptions in a way that all these conditions are satisfied. A related analysis of invariant curves and their applications for NODEs have been presented in [19]. In [19] invariant curves have been applied to the given exact form of different symmetries. But in this paper, we have demonstrated how to obtain the exact form of symmetries using the Fundamental theorem of Lie in example 2.8 under theorem 5. And we have combined the fundamental theorem of Lie , the Linearized symmetry condition and the idea of invariant curves to present a complete solution method. In [19] invariant solution curves have been illustrated with the family of solution curves. In the current project, we plotted the obtained invariant solution curves with the related symmetries(flow curves) and their vector fields, in Figure 2and Figure 2. From Figure 1 and Figure 2, we have graphically illustrated how some particular solution curves of an ODE exit in a way that no flow curves cross them, which shows the underlying meaning theorem 2.12.

In examples 6.5 and 6.4 order of the seconded order ODEs have been reduced using canonical coordinates of Lie symmetry groups. If the reduced second Order NODE is a first-order NODE, then we are able to use one of the first two methods to solve that NODE as we demonstrated in example 6.5 . Similarly, we can reduce the order of third or higher-order NODEs, since the order reduction of NODE has been defined for any order NODE. Then, if the reduced NODE is also a NODE, we are able to go towards the exact solution using these given three methods. In this project Maple software has been used to solve determining equations in examples 6.5 and 6.4.

It is obvious that the main challenge of applying these methods is to find the appropriate assumptions for the form of infinitesimals. Although we are able to use the PDEtools package on MAPLE to check whether the assumptions are correct, finding an assumption that satisfies all the conditions under method 2, is not an easy task in general.

8 Conclusion

NODEs are widely used in many disciplines and approximation methods are used to get the approximated solutions due to the lack of techniques to get their general solutions. Therefore the main objective of this project is to present how to obtain exact solutions of NODEs using their Lie symmetries. In examples 6.1,6.1.1 and 6.2 we have demonstrated how to use symmetries of first-order NODEs to successfully obtain their general solutions under two different methods, but the second method in which we use invariant cures is generally more hard and complex task than method1 due to the requirement of different sets of infinitesimals. In examples 6.5 and 6.4 we have demonstrated how to reduce the order of second-order NODE using the prolongation of Lie symmetries. Using these three methods we are able to solve the third or higher-order NOSEs. In all these three methods, the standard types of ODEs have not been taken into consideration. Therefore we are able to apply these methods for any unfamiliar type of ODE which is not fit into standard ODE types such as exact, homogeneous, separable, etc. But finding infinitesimals may be a challenging task for some ODEs. Thus we can consider these symmetry methods as powerful solution techniques for solving NODEs, which give exact general solutions. This project can be extended for finding exact solutions for nonlinear partial differential equations using Lie Algebra.

9 Declarations

9.1 Competing Interests

There are no conflicts of interest that are relevant to this study for the authors of this manuscript.

9.2 Publisher's Note

AIJR remains neutral with regard to jurisdictional claims in published institutional affiliations.

How to Cite this Article:

D. H. S. Perera and D. Gallage, "Solution Methods for Nonlinear Ordinary Differential Equations Using Lie Symmetry Groups", *Adv. J. Grad. Res.*, vol. 13, no. 1, pp. 37–61, Feb. 2023. doi: 10.21467/ajgr.13.1.37-61

References

[1] P. J. Olver, *Applications of Lie groups to differential equations*, vol. 107. Springer Science & Business Media, 2000.

[2] M. R. Ali, W.-X. Ma, and R. Sadat, "Lie symmetry analysis and wave propagation in variable-coefficient nonlinear physical phenomena," *East Asian J. Appl. Math.*, vol. 12, no. 1, pp. 201–212, 2022.

[3] S. Kumar and S. K. Dhiman, "Lie symmetry analysis, optimal system, exact solutions and dynamics of solitons of a (3+ 1)-dimensional generalised bkp–boussinesq equation," *Pramana*, vol. 96, no. 1, p. 31, 2022.

[4] D. V. Tanwar and M. Kumar, "On lie symmetries and invariant solutions of broer–kaup–kupershmidt equation in shallow water of uniform depth," *Journal of Ocean Engineering and Science*, 2022.

[5] N. Raza, F. Salman, A. R. Butt, and M. L. Gandarias, "Lie symmetry analysis, soliton solutions and qualitative analysis concerning to the generalized q-deformed sinh-gordon equation," *Communications in Nonlinear Science and Numerical Simulation*, vol. 116, p. 106824, 2023.

[6] S.-F. Tian, "Lie symmetry analysis, conservation laws and solitary wave solutions to a fourth-order nonlinear generalized boussinesq water wave equation," *Applied Mathematics Letters*, vol. 100, p. 106056, 2020.

[7] L. Xia and X. Ge, "Lie symmetry analysis and conservation laws of the axially loaded euler beam," *Mathematics*, vol. 10, no. 15, p. 2759, 2022.

[8] A. Paliathanasis, "Lie symmetries and similarity solutions for rotating shallow water," *Zeitschrift für Naturforschung A*, vol. 74, no. 10, pp. 869–877, 2019.

[9] D. H. Sattinger and O. L. Weaver, *Lie groups and algebras with applications to physics, geometry, and mechanics*, vol. 61. Springer Science & Business Media, 2013.

[10] P. E. Hydon, "Introduction to symmetry methods in the solution of differential equations that occur in chemistry and chemical biology," *International journal of quantum chemistry*, vol. 106, no. 1, pp. 266–277, 2006.

[11] M. Edwards and M. Nucci, "Application of lie group analysis to a core group model for sexually transmitted diseases," *Journal of Nonlinear Mathematical Physics*, vol. 13, no. 2, pp. 211–230, 2006.

[12] V. Torrisi and M. Nucci, "Application of lie group analysis to a mathematical model," in *The Geometrical Study of Differential Equations: NSF-CBMS Conference on the Geometrical Study of Differential Equations, June 20-25, 2000, Howard University, Washington*, vol. 285, p. 11, American Mathematical Soc., 2001.

[13] M. B. Matadi, "Application of lie symmetry to a mathematical model that describes a cancer subnetwork," *Symmetry*, vol. 14, no. 2, p. 400, 2022.

[14] M. Nuccia and P. Leach, "Singularity and symmetry analyses of mathematical models of epidemics," *South African Journal of Science*, vol. 105, no. 3, pp. 136–146, 2009.

[15] J. Yu, Y. Feng, and X. Wang, "Lie symmetry analysis and exact solutions of time fractional black–scholes equation," *International Journal of Financial Engineering*, vol. 9, no. 04, p. 2250023, 2022.

[16] S. Kontogiorgis and C. Sophocleous, "Lie symmetries and the constant elasticity of variance (cev) model," *Partial Differential Equations in Applied Mathematics*, vol. 5, p. 100290, 2022.

[17] D. J. Arrigo, *Symmetry analysis of differential equations: an introduction.* John Wiley & Sons, 2015.

[18] P. J. Olver, "Nonlinear ordinary differential equations."

[19] F. Granström, "Symmetry methods and some nonlinear differential equations: Background and illustrative examples," 2017.

[20] P. E. Hydon and P. E. Hydon, *Symmetry methods for differential equations: a beginner's guide*. No. 22, Cambridge University Press, 2000.

[21] G. W. Bluman and S. Kumei, *Symmetries and differential equations*, vol. 81. Springer Science & Business Media, 2013.

[22] P. Etingof, "Lie groups and lie algebras," *arXiv preprint arXiv:2201.09397*, 2022.

[23] D. H. S. Perera and D. Gallage, "Lie groups of symmetries for nonlinear ordinary differential equations,"

[24] N. Dehmamy, R. Walters, Y. Liu, D. Wang, and R. Yu, "Automatic symmetry discovery with lie algebra convolutional network," *Advances in Neural Information Processing Systems*, vol. 34, pp. 2503–2515, 2021.

[25] G. Miller, "An introduction to the lie theory of one-parameter groups, with applications to the solution of differential equations. by abraham cohen, ph. d., associate in mathematics, johns hopkins university. boston, dc heath & co. 1911. pp. iv+ 248. half leather.," *Science*, vol. 34, no. 887, pp. 924–924, 1911.

[26] A. Hussain, S. Bano, I. Khan, D. Baleanu, and K. Sooppy Nisar, "Lie symmetry analysis, explicit solutions and conservation laws of a spatially two-dimensional burgers–huxley equation," *Symmetry*, vol. 12, no. 1, p. 170, 2020.

[27] J. Starrett, "Solving differential equations by symmetry groups," *The American Mathematical Monthly*, vol. 114, no. 9, pp. 778–792, 2007.

[28] E. Cheb-Terrab, L. Duarte, and L. Da Mota, "Computer algebra solving of first order odes using symmetry methods," *Computer physics communications*, vol. 101, no. 3, pp. 254–268, 1997.

[29] E. Cheb-Terrab, L. Duarte, and L. Da Mota, "Computer algebra solving of second order odes using symmetry methods," *Computer Physics Communications*, vol. 108, no. 1, pp. 90–114, 1998.

[30] M. Hassan, A. Shehata, and M. Abdel-Daym, "The investigation of exact solutions and conservation laws of the classical boussinesq system via the lie symmetry method," *Applied. Mathematics and Information Science*, vol. 16, no. 2, pp. 177–185, 2022.

[31] S. Kumar and I. Hamid, "Dynamics of closed-form invariant solutions and diversity of wave profiles of (2+ 1)-dimensional ito integro-differential equation via lie symmetry analysis," *Journal of Ocean Engineering and Science*, 2022.

[32] K. Bibi and K. Ahmad, "Exact solutions of newell-whitehead-segel equations using symmetry transformations," *Journal of Function Spaces*, vol. 2021, pp. 1–8, 2021.

[33] J. Borgqvist, F. Ohlsson, and R. E. Baker, "Symmetries of systems of first order odes: symbolic symmetry computations, mechanistic model construction and applications in biology," *arXiv preprint arXiv:2202.04935*, 2022.

[34] C. N. Retz, *Classical and Nonclassical Lie Symmetries of the K (m, N) Dispersion Equation*. PhD thesis, University of North Carolina Wilmington, 2012.