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Remarks on the Solution of Fractional Ordinary Differential Equations Using Laplace Transform Method

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ABSTRACT

In this work we used the Laplace transform method to solve linear fractional-order differential equation, fractional ordinary differential equations with constant and variable coefficients. The solutions were expressed in terms of Mittag-Leffler functions, and then written in a compact simplified form. As a special case for simplicity, the order of the derivative determined the order of the solution that was obtained. This paper presented several case studies involving the implementation of Fractional Order calculus-based models, whose results demonstrate the importance of Fractional Order Calculus.

Keywords: Fractional calculus; Laplace transform; Mittag-Leffler function

1 Introduction

Generalized integro-differentiation appears to be a better name, but fractional calculus (FC) persisted for historical reasons [1]. The FC extends the standard differential calculus to non-integer orders, whether real or complex. Until the past few decades, when the research community began to notice its excellent performance for describing a wide range of natural and artificial processes, this scientific tool was mostly used in pure mathematics. Recent trends in FC and a thorough presentation of current knowledge can be found in [16]. Physical phenomena can be articulated with the aid of the theory of fractional order derivatives and integrals, and fractional techniques can also successfully simulate real-life phenomena that depend not only on the present but also on the past time history [1]. Therefore, several methods [2, 3, 4, 5, and 10] are still being developed to solve fractional differential equations in order to achieve an exact and numerical solution. Fractional notions have been employed as a tool in domains including engineering, economics, physics, and chemistry. Today, research and development on fractional calculus are being applied to the study of differential equations, enabling the ordering of both ordinary and partial differential equations by any number [1]. The applications of fractional differential equations in fields including biology, economics, the oil industry, finance, engineering, and a wide range of other fields have been the primary drivers of research in this field [6, 8, 9, 11, 12, 13, 14 and 15]. In this study, we solve linear fractional ordinary differential equations with constant and variable coefficients using the Laplace transform. The outcomes are then simplified and stated in terms of Mittag-Leffler functions.

2 Materials and Methods

For the assessment of the fractional calculus [1,2,3, and 7] that will be employed in this study, a few definitions and mathematical foundations are presented in this section. Mathematically, the Riemann-Liouville fractional integral of order a is defined as:

2.1 The Riemann-Liouville integral is defined as

$$I^{a} f(x) = \frac{1}{\Gamma(a)} \int_{a}^{x} f(t)(x-t)^{a-1} dt \tag{1}$$

2.2 Laplace Transform of the Fractional Integral

2.2.1 Laplace Transform

The Laplace transform of a function $\mathrm{f(t)}$, denoted by $\mathrm{F(s)}$, is defined by the equation

$$F(s) = (L\, f\,)(s) = L\{\, f\,(t);\, s\} = \int_{0}^{\infty} f(t) e^{-st} dt \tag{2}$$

$$_{0}I_{x}^{-\alpha} f(x) = \frac{1}{\Gamma(a)} \int_{a}^{x} f(t)(x-t)^{a-1} dt \tag{3}$$

Application of convolution theorem of the Laplace transform gives

$$L\{\, _{0}I_{x}^{-\alpha} f(x); s\} = L\{\frac{t^{\alpha-1}}{\Gamma(a)}\}\, L\{f(t); s\} = s^{-\alpha} F(s) \tag{4}$$

2.3 Caputo derivative

If m is the smallest integer greater than α, then Caputo fractional derivative of order α >0 is defined as

$D_{*}^{\alpha} f(x) = J^{m-\alpha} f^{m}(x)$ with $m-1 < \alpha < m$,

given

$$D_{*}^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \left[\int_{0}^{x} \frac{f^{(m)}(\tau)}{(x-\tau)^{\alpha+1-m}} d\tau \right], & m-1 < \alpha < m, \\ \frac{d^{m}}{dx^{m}} f(x), & \alpha = m. \end{cases} \tag{5}$$

2.4 Laplace transform of Caputo fractional derivative

$$_{a}D_{x}^{\alpha} f(x) = \, _{a}I_{x}^{n-\alpha} \frac{d^{n}}{dx^{n}} f(x) = \, _{a}D_{t}^{-(n-\alpha)} f^{n}(t) \tag{6}$$

Mamman *et al.*, *Adv. J. Grad. Res.;* Vol. 14, Issue 1, pp: 21-26, January 2024

2.5 Mittag-Leffler Function

The special function of Mittag-Leffler for $\alpha, \beta \in \mathbb{C}, Re(\alpha), Re(\beta) > 0$ is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{Z^k}{\Gamma(\alpha k+\beta)} \tag{7}$$

The function $E(t, \alpha, a)$ is used to solve differentials equations of fractional order which is defined by:

$$E(t, \alpha, a) = t^{\alpha} \sum_{k=0}^{\infty} \frac{(at)^k}{\Gamma(k+\alpha+1)} = t^{\alpha} E_{1,\alpha+1}(at) \tag{8}$$

Property 1: $E(t, \alpha, a) = \frac{1}{\Gamma(\alpha)} \int_0^t \xi^{\alpha-1} e^{a(t-\xi)} d\xi$

Property 2: $\mathcal{L}^{-1}\left[\frac{S^{-(\alpha-\beta)}}{S^{\beta}-a}\right] = t^{\alpha-1}\beta_{\alpha}(at^{\beta}), \qquad |S^{\beta} - a| < 1$

Corollary 1:

i. $E_{1,\frac{3}{2}}(at) = \frac{e^{at}}{\sqrt{at}} \operatorname{erf}(\sqrt{at})$

ii. $E_{1,\frac{1}{2}}(at) = \frac{1}{\sqrt{\pi}} + \sqrt{at} e^{at} erf(\sqrt{at})$

iii. $E_{1,\frac{5}{2}}(at) = \frac{1}{at}\left[\frac{e^{at}}{\sqrt{at}} \operatorname{erf}(\sqrt{at}) - \frac{2}{\sqrt{\pi}}\right]$

iv. $E_{1,-\frac{1}{2}}(at) = \frac{-1}{2\sqrt{\pi}} + (at)(\frac{1}{\sqrt{\pi}} + \sqrt{at} e^{at} \operatorname{erf}(\sqrt{at}))$

Corollary 2:

i. $\mathcal{L}^{-1}\left[\frac{1}{S^{\alpha}(S-a)^2}\right] = tE(t, \alpha, a) - \alpha E(t, \alpha + 1, a)$

ii. $\mathcal{L}^{-1}\left[\frac{1}{S^{\alpha}(S-a)^3}\right] = \frac{1}{2} t^2 E(t, \alpha, a) - \alpha t E(t, \alpha + 1, a) + \frac{\alpha(\alpha+1)}{2} E(t, \alpha + 2, a)$

3 Result and Discussion

Example 1: Consider the following initial value problem in the case of the inhomogeneous Bagley-Tonick equation

$$D^2 y(x) + D^{\frac{3}{2}}(x) + y(x) = 1 + x$$

$$y(0) = y'(0) = 1$$

Using Laplace transform

$$s^2F(s) - sy(0) - y'(0) + \frac{s^2F(s) - sy(0) - y'(0)}{s^{\frac{1}{2}}} + F(s) = \frac{1}{s} + \frac{1}{s^2}$$

Considering the condition

$$y(0) = y'(0) = 1$$

We have

$$F(s) = \left(\frac{1}{s} + \frac{1}{s^2}\right)$$

Example 2: This problem covers the inhomogeneous linear equation

$$D^{\alpha}y(x)+y(x)=\frac{2x^{2-\alpha}}{\Gamma(3-\alpha)}-\frac{x^{1-\alpha}}{\Gamma(2-\alpha)}+x^2-x$$
$$y(0)=0, 0<\alpha\leq 1$$

Using the Laplace transform, $F(s)$ is obtained as follows

$$\frac{sF(s)-y(0)}{s^{1-\alpha}}=\frac{2}{s^{3-\alpha}}-\frac{1}{s^{2-\alpha}}-F(s)+\frac{2}{s^3}-\frac{1}{s^2}$$
$$F(s)=\frac{2}{s^3}-\frac{1}{s^2}$$

Example 3: Consider the following fractional ordinary differential equation with variable coefficients

$tD^{\alpha}x(t)+D^{\alpha-1}x(t)+tx(t)=0 \qquad x(0)=1,\ 1<\alpha\leq 2$

Application of Laplace transform gives

$$-\frac{\mathrm{d}}{\mathrm{ds}}\mathcal{L}\{tD^{\alpha}x(t)\}+\mathcal{L}\{D^{\alpha-1}x(t)\}+\mathcal{L}\{tx(t)\}=0$$
$$\Rightarrow -\frac{d}{ds}\left[s^{\alpha}\bar{x}(s)-\sum_{k=0}^{1}s^kD^{\alpha-k-1}x(0)\right]+\left[s^{\alpha-1}\bar{x}(s)-\sum_{k=0}^{0}s^kD^{\alpha-k-2}x(0)\right]$$
$$-\frac{d\bar{x}(s)}{ds}=0$$
$$\therefore x(t)=\mathcal{L}^{-1}\left[c(1+s^{\alpha})\frac{(1-\alpha)}{\alpha}\right]$$

As special case we take $\alpha=2$ then we have

$$x(t)=\mathcal{L}^{-1}\left[\frac{c}{\sqrt{1+s^2}}\right]=cJ_0(t)$$

Example 4: Consider the homogenous equation.

$$\left(D^1-3D^{\frac{1}{2}}+2D^0\right)x(t)=0$$

Solution

By applying Laplace transform we have

$$\mathcal{L}\{Dx(t)\}-3\mathcal{L}\left\{D^{\frac{1}{2}}x(t)\right\}+2\mathcal{L}\{D^0x(t)\}=0$$
$$s\bar{x}(s)-x(0)-3\mathcal{L}\left\{D\left(D^{-\frac{1}{2}}x(t)\right)\right\}+2\bar{x}(s)=0$$
$$c\left(\frac{2}{s-4}+\frac{1}{s^{-\frac{1}{2}}(s-4)}-\frac{1}{s-1}-\frac{1}{s^{-\frac{1}{2}}(s-1)}\right)$$

Where

$$c=\left[x(0)-3D^{-\frac{1}{2}}x(0)\right]$$

Mamman *et al., Adv. J. Grad. Res.;* Vol. 14, Issue 1, pp: 21-26, January 2024

From corollary (1) we obtain the solution as follows

$$\therefore x(t) = c\left[2e^{4t} - e^{t} + E\left(t, -\frac{1}{2}, 4\right) - E\left(t, -\frac{1}{2}, 1\right)\right]$$
$$c\left[2e^{4t}\, erfc\left(-2\sqrt{t}\right) - e^{t}\left(erf\ c\left(-\sqrt{t}\right)\right)\right]$$

Example 5: Consider the inhomogeneous initial value problem.

$$\left(D^{1} - 2D^{\frac{1}{2}} + D^{0}\right)x(t) = e^{t} \qquad x(0) = 1$$

Solution

We applied the Laplace transform to obtain:

$$s\bar{x}(s) - x(0) - 2s^{\frac{1}{2}}\bar{x}(s) + 2x(0) + \bar{x}(s) = \frac{1}{s-1}$$

From property 2, and corollary 2 we have

$$\mathrm{x(t)} = \frac{1}{4}e^{t} + \left(c + \frac{3}{4}\right)\mathcal{L}^{-1}\left\{\frac{1}{s^{-1}(s-1)^{2}}\right\} + \left(\frac{3}{2} + 2c\right)\mathcal{L}^{-1}\left\{\frac{1}{s^{-\frac{1}{2}}(s-1)^{2}}\right\}$$
$$\left(\frac{3}{4} + c\right)\mathrm{t}e^{t} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^{-\frac{3}{2}}(s-1)^{3}}\right\} + \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^{-1}(s-1)^{2}}\right\}$$
$$\frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^{-\frac{1}{2}}(s-1)^{3}}\right\} + \frac{1}{2}t^{2}e^{t}$$

Applying the initial condition $x(0) = 1$ gives $\mathrm{c} = 0$ and by substituting the value of c we have:

$$x(t) = \frac{e^{t}}{4}(3t^{2} + 12t + 4) + \frac{\sqrt{t}}{4}(9 + 3t)E_{1,\frac{1}{2}}(t) + \frac{\sqrt{t}}{16}(15 + 12t)E_{1,\frac{3}{2}}(t) + \frac{\sqrt{t}}{4}E_{1,-\frac{1}{2}}(t)$$
$$-\frac{3}{16}t^{\frac{3}{2}}E_{1,\frac{5}{2}}(t)$$

from corollary 2 we get the final solution as follows

$$x(t) = \frac{1}{16}\left\{4(3t^{2} + 12t + 4)e^{t} + (15 + 48t + 16t^{2} - 3t^{3})e^{t}erf\left(\sqrt{t}\right) + 2\sqrt{\frac{t}{\pi}}\left(3t^{\frac{3}{2}} + 8t + 17\right)\right\}$$

4 Conclusion

The Laplace transformation method has been successfully applied to find the exact solution of linear fractional ordinary differential equations also fractional ordinary differential equations, with variable coefficients. Without assumptions, we applied our method directly. In this paper, fractional order calculus is treated more suggestively rather than rigorously. The examples presented show the effectiveness of Laplace transform approach of solving Fractional Order calculus-based models, whose results demonstrate the importance of Fractional Order Calculus.

5 Declarations

5.1 Competing Interests

The authors have no conflicts of interest to declare that are relevant to the content of this article.

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